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Double-diffusive boundary layers along vertical free surfaces

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Abstract—This paper deals with double-diffusive (or thermosolutal) combined free convection, i.e. free convection due to buoyant forces (natural convection) and surface tension gradients (Marangoni convection), which are generated by volume differences and surface gradients of temperature and solute concentration. Attention is focused on boundary layers that form along a vertical liquid–gas interface, when the appropriately defined non-dimensional characteristic transport numbers are large enough, in problems of thermosolutal natural and Marangoni convection, such as buoyancy and surface tension driven flows in differentially heated open cavities and liquid bridges. Classes of similar solutions are derived for each class of convection on the basis of a rigorous order of magnitude analysis. Velocity, temperature and concentration profiles are reported in the similarity plane; flow and transport properties at the liquid–gas interface (interfacial velocity, heat and mass transfer bulk coefficients) are obtained for a wide range of Prandtl and Schmidt numbers and different values of the similarity parameter.

1. INTRODUCTION

IN THE absence of imposed velocities and pressure differences, convection in systems with fluid–fluid interfaces may be due to volume driving forces (buoyancy) and surface driving forces (Marangoni stresses). Volume driving forces act in the direction of the gravitational vector (**g**) and are proportional to density differences. Marangoni stresses act on the interface and are due to surface gradients of the interfacial tension (σ). Both may be generated by volume differences and surface gradients of temperature and/or solute concentration.

In this paper we consider interface dissipative flows of the boundary layer (BL) type, i.e. thin dissipative layers that may form near free surfaces when the appropriately defined transport numbers (Reynolds, Peclet, etc.) are large enough [1].

Interface boundary layers will be called Marangoni boundary layers (MBLs), natural (or buoyant) boundary layers (NBLs) or combined boundary layers (CBLs), according to whether the driving actions are due to Marangoni stresses, only, to buoyant forces only or to both. A further classification is related to the nature (thermal or solutal) of the driving actions. A boundary layer will be called 'singly-diffusive' if all driving actions are of the same nature (thermal *or* solutal) or 'double-diffusive' if the driving actions are of a different nature (thermal *and* solutal).

The theory of MBLs has been formulated, up until now, only in the case of thermal free convection, so that only singly-diffusive, thermal MBLs have been investigated. Napolitano [1] carried out a systematic derivation of the steady thermal MBL equations and outlined a priori criteria for their applicability. Plane thermal MBLs were analysed by Napolitano and Golia [2], who addressed the question of the existence and characterization of similar solutions, found similarity classes and obtained numerical solutions both in the case of uncoupled and coupled flow fields of the two interfacing fluids.

In the case of axial-symmetric MBLs [3], by applying the Mangler transformation, the field equations assume the same form as for plane motion, except for the interface tangential momentum balance equation, in which the curvature radius of the interface explicitly appears. In this case similar solutions exist only if the equation of the interface is described by a power law in the Mangler variable.

Natural boundary layers (NBLs) along fluid–fluid interfaces have been investigated only for the freely rising plume flow [4] when volume driving forces (buoyancy) of different nature (thermal and solutal) are present (double-diffusive NBLs). Similar solutions of thermal CBLs have been analysed by Golia and Viviani [5, 6].

Recently Batishchev [7] reviewed MBLs considering mainly the work done in the Soviet literature. Addition to the paper of Batishchev and more details on future research topics about MBLs can be found in the review paper by Napolitano and Viviani [8].

Here we present a mathematical formulation for the general case of interface boundary layers due to volume and surface driving actions, both generated by differences and surface gradients of temperature and/or solutal concentration. The classes of Marangoni and combined boundary layers include, as particular cases, those already investigated of thermal MBL [2] and CBL [5]. The class of NBLs, analysed here for different values of the similarity parameter in both cases of singly-diffusive and double-diffusive boundary layers, includes the case of plume flows considered in ref. [4] (exponents of the power laws for the temperature and concentration equal to $-3/5$) in their

NOMENCLATURE

c	normalized solutal concentration
c_0	concentration scale factor
C	solutal concentration
D	mass diffusivity
f	stream function in the similarity plane
g	temperature in the similarity plane
$\mathbf{g}$	gravity vector
g_x	component of the gravity vector along x
G_C^*	solutal Grashof number
G_T^*	thermal Grashof number
h	solutal concentration in the similarity plane
$\underline{k}$	non-dimensional curvature tensor of the interface
$\underline{K}$	curvature tensor of the interface
$l(x)$	local normal scale factor
l_0	constant normal scale factor
M_C^*	solutal Marangoni number
M_T^*	thermal Marangoni number
$\mathbf{n}$	unit normal to the interface
P	pressure
Pe_C^*	solutal Peclet number
Pe_T^*	thermal Peclet number
Pr	Prandtl number
Re^*	Reynolds number
Sc	Schmidt number
t	non-dimensional temperature
t_0	temperature scale factor
T	temperature
ΔT	imposed temperature difference
u	non-dimensional velocity component in the x-direction
v	non-dimensional velocity component in the y-direction
$\mathbf{v}$	non-dimensional velocity vector
$\mathbf{V}$	velocity vector
$\mathbf{V}_t$	component of velocity tangent to the interface
V_D	mass diffusion speed
V_g	buoyant speed
V_m	Marangoni speed
V_r	reference velocity
V_α	thermal diffusion speed
V_ν	momentum diffusion speed
x	non-dimensional arc length along the interface
y	non-dimensional Euclidean distance from the interface.
Greek symbols	
α	thermal diffusivity
β	power law exponent for the temperature
β_T, β_C	coefficient of expansion with temperature and solute concentration
γ	power law exponent for the concentration
δ	jump operator
δ^*	boundary layer displacement thickness
η	similarity variable
λ	power law exponent for the normal scale factor
μ	dynamic viscosity
ν	momentum diffusivity
π	non-dimensional pressure
ρ	density
σ	equilibrium surface tension
σ_T, σ_C	rates of change of surface tension with temperature and solute concentration
ψ	non-dimensional stream function
ω	power law exponent for the pressure
∇	nabla operator.

analysis of boundary layers along vertical solid walls.

The paper runs as follows: the basic assumptions for the problem under study and the corresponding field equations are presented in Section 2; the non-dimensionalization process is then performed in Section 3; Section 4 deals with the search and characterization of similar solutions, according to a rigorous and coherent order of magnitude analysis. Such a unitary approach determines, at the same time, the a priori conditions, expressed in terms of the problem's data, for the existence of the different BL regimes (Marangoni, natural and combined; thermal, solutal, thermosolutal) and the conditions for the existence of similar solutions. All possible cases are presented in Section 5, where the similarity conditions, the pertinent expressions for the scale factors, and the field equations are discussed for each specific BL regime. Numerical results are presented in Section 6. Final comments and conclusions are outlined in the last section.

2. BASIC ASSUMPTIONS AND FIELD EQUATIONS

The geometry of the problem is shown in Fig. 1. We shall consider steady, plane flows in the boundary layer along the liquid side of the interface Σ of a plane liquid bridge, of extension L, parallel to the gravitational vector $\mathbf{g}$; X and Y are dimensional Cartesian coordinates; x and y are non-dimensional boundary layer coordinates with x the arc length along Σ, and y the Euclidean distance from the interface Σ. The end walls are maintained at a given positive temperature difference $\Delta T = T_H - T_C$ and there are neither imposed velocities nor imposed pressure differences.

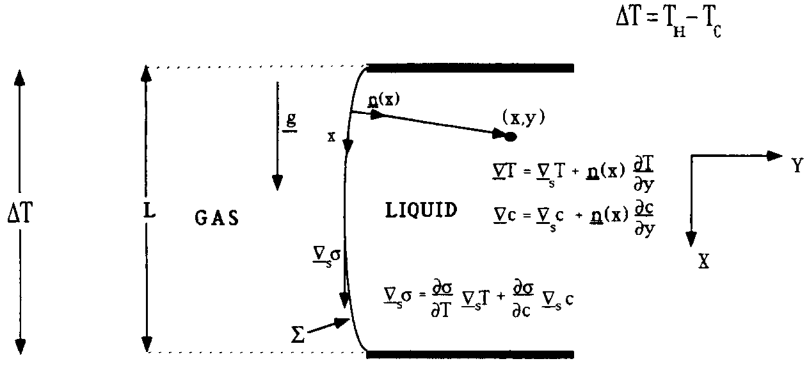


FIG. 1. Geometry of the problem.

The liquid is a Newtonian ideal binary mixture with constant diffusion coefficients; Soret and Dufour effects will be neglected; the Boussinesque approximation applies; viscous dissipation is negligible. Free convective motion is induced by Marangoni stresses and buoyancy forces arising from temperature and solute concentration gradients and/or differences. The interface Σ is modelled as a massless stream-surface with two intensive degrees of freedom (temperature and electrochemical potential), in thermal, mechanical and chemical equilibrium with the adjacent volume phases [9].

Momentum, energy and mass coupling between liquid and gas phases, radiation effects and surface irreversibility are assumed to be negligible.

The relevant state equations for the volume and surface phases are assumed to be given, with the above-mentioned thermal and chemical equilibrium hypothesis of the surface phase, by

$$\rho = \rho_h[1-\beta_T(T-T_h)-\beta_C(C-C_h)] \qquad (1a)$$

$$\sigma = \sigma_h-\sigma_T(T-T_h)-\sigma_C(C-C_h) \qquad (1b)$$

where ρ is the density, σ the equilibrium surface tension, $\rho_h\beta_T$, $\rho_h\beta_C$ and σ_T, σ_C the constant rates of change with temperature T and solute concentration C of density and surface tension, respectively. The subscript (h) denotes values pertaining to the hydrostatic state, assumed uniform and chosen as the reference state.

The curvilinear coordinate x is always directed in the sense of the motion. The driving forces, for the problem under study, are the buoyant force, $(g/\rho_h)\nabla\rho$, in the volume phase, and the surface gradient of σ, on the liquid–gas interface, which, upon equations (1a) and (1b), are given by

$$(g/\rho_h)\nabla\rho = -g(\beta_T\nabla T+\beta_C\nabla C) \qquad (2a)$$

$$\nabla_s\sigma = -\sigma_T\nabla_s T-\sigma_C\nabla_s C \qquad (2b)$$

where ∇_s denotes the surface gradient operator [10].

The directions of the driving actions depend on the orientation of the temperature and concentration gradients in the liquid, and on the signs of the thermodynamic coefficients σ_T, σ_C, β_T, β_C. There are several combinations of the signs and magnitudes of ∇T and ∇C, and of the thermodynamic coefficients σ_T, σ_C, β_T, β_C, which determine the direction of the motion in the physical plane. The mathematical formulation we shall derive in the next sections will be aimed at minimizing the number of solutions, in the similarity plane, able to cover all possible cases in the physical plane. This will be done by using the criterion of maximum normalization of the balance equations and of the interface boundary conditions by means of scale factors, for the flow field quantities, the signs and values of which account for the effective physical situation, i.e. signs and magnitudes of ∇T and ∇C, σ_T, σ_C, β_T, β_C.

With the above assumptions and positions the field equations are the thermodynamic state equations (1a) and (1b), the balance equations for the liquid and the balance equations for the surface phase, representing the boundary conditions for the liquid balance equations

$$\nabla\cdot\mathbf{V} = 0 \qquad (3a)$$

$$\mathbf{V}\cdot\nabla\mathbf{V}+\frac{1}{\rho_h}\nabla P = \nu\nabla^2\mathbf{V}+\frac{\rho}{\rho_h}\mathbf{g} \qquad (3b)$$

$$\mathbf{V}\cdot\nabla T = \alpha\nabla^2 T \qquad (3c)$$

$$\mathbf{V}\cdot\nabla C = D\nabla^2 C \qquad (3d)$$

$$\mathbf{n}\cdot\mathbf{V} = 0 \qquad (4a)$$

$$\delta(\mathbf{V}_t) = 0, \quad \mathbf{V}_t = (\mathbf{n}\wedge\mathbf{V})\wedge\mathbf{n} \qquad (4b)$$

$$\delta[\mu(\mathbf{n}\cdot\nabla\mathbf{V}_t-\mathbf{V}_t\cdot\underline{K})]+\nabla_s\sigma = 0 \qquad (4c)$$

$$\delta T = 0 \qquad (4d)$$

$$\delta C = 0 \qquad (4e)$$

where $\delta f = f(x,0^+)-f(x,0^-)$ is the jump of f at the interface, $\mathbf{n}$ its unit normal oriented toward the liquid (see Fig. 1), $\underline{K}$ the curvature tensor of the interface Σ; P the pressure; μ the dynamic viscosity; ν, α and D momentum, thermal and solute mass diffusivities, and all other symbols have already been defined.

3. NON-DIMENSIONAL FIELD EQUATIONS

3.1. *General remarks*

To put the field equations in the correct non-dimensional form, the following facts must be taken into account.

(a) The set of reference quantities cannot be chosen arbitrarily a priori but must be determined, in terms of the problem's data only, from a rigorous and systematic order of magnitude analysis (OMA) of the field equations.

(b) When dissipative layers along the interface are present, the flow field is not isotropic. A physically correct non-dimensional form of the field equations must exhibit this anisotropy so that, in particular, reference quantities for velocity and length in directions normal and parallel to the interface must be allowed to be of different orders of magnitude (anisotropic reference quantities for vectors and tensors).

(c) The coordinate system has to single out explicitly the direction field of the normal **n** to the interface and the most natural choice is the parallel-surface coordinate system (pscc) [10, 11], in which one of the coordinates is the Euclidean distance from Σ and the other two are arbitrary curvilinear coordinates on Σ. In the subject case, the problem's data are appropriately combined in the set of seven characteristic speeds defined by

$$V_\nu = \frac{\nu}{L}, \quad V_\alpha = \frac{\alpha}{L}, \quad V_D = \frac{D}{L} \qquad (5a–c)$$

$$V_{g_T} = \frac{g_x\beta_T(\Delta T)L^2}{\nu}, \quad V_{g_C} = \frac{g_x\beta_C(\Delta C)L^2}{\nu} \qquad (6a, b)$$

$$V_{m_T} = \frac{\sigma_T(\Delta T)}{\mu}, \quad V_{m_C} = \frac{\sigma_C(\Delta C)}{\mu} \qquad (7a, b)$$

and referred to as 'diffusion' speeds (5a)–(5c), 'volume (or buoyant) driving' speeds (6a) and (6b) and 'surface (or Marangoni) driving' speeds (7a) and (7b). In (6a) and (6b) g_x denotes the component of the gravity vector along **x**. The signs of the driving speeds depend on those of the thermodynamic derivatives involved in their definition. The values of the volume driving speeds (6a) and (6b) may be limited by stability considerations.

The ratios (L_{r_n}/L) and (V_{r_n}/V_r) between reference lengths and velocities in directions normal and parallel to the interface are of the same order, upon the continuity equation (3a). This common order of magnitude will be referred to as the 'normal scale factor' and denoted by (l). In principle, reference quantities can be either constant (global non-dimensional formulation) or functions of the coordinate on the surface Σ (local non-dimensional formulation). Their determination, on the basis of a rigorous OMA of the field equations, leads to a constrained maximum problem formulated, analysed and solved by Napolitano [12] for the case of constant reference quantities.

In this paper we present an extension of Napolitano's constrained problem to the case in which reference quantities are variable (local non-dimensional formulation). After having formulated the vectorial non-dimensional form of the field equations by means of constant and isotropic reference quantities, we shall perform, simultaneously, the appropriate order of magnitude analysis and the search for similarity conditions by introducing variable, non-isotropic, reference quantities. Such a unitary approach will determine, at the same time, the a priori conditions, expressed in terms of the problem's data, for the existence of the different BL regimes (Marangoni, natural and combined) and the conditions under which similarity prevails. The analysis will be limited to first order in the asymptotic expansions. Second-order boundary layer effects will be considered in future papers. The class of variation considered is the power class: extension to the other class, such as the exponential one, should be straightforward and will not be considered in this paper.

3.2. *Constant and isotropic reference quantities*

Since there is no imposed pressure difference we let

$$\mathbf{V} = V_r\mathbf{v}, \quad \nabla = \frac{1}{L}\nabla^* \qquad (8a, b)$$

$$P = P_h + \rho_h V_r^2 \pi, \quad T = T_h + (\Delta T)t \qquad (8c, d)$$

$$C = C_h + (\Delta C)c \qquad (8e)$$

where, as said, ΔT is the imposed positive temperature difference (Fig. 1), and ΔC the characteristic positive concentration difference.

The non-dimensional equations and boundary conditions for the liquid volume phase read

$$\nabla^* \cdot \mathbf{v} = 0 \qquad (9a)$$

$$Re^*[\mathbf{v}\cdot\nabla^*\mathbf{v} + \nabla^*\pi] = \nabla^{*2}\mathbf{v} - (G_T^* t + G_C^* c)\mathbf{i}_x \qquad (9b)$$

$$Pe_T^* \mathbf{v}\cdot\nabla^* t = \nabla^{*2} t \qquad (9c)$$

$$Pe_C^* \mathbf{v}\cdot\nabla^* c = \nabla^{*2} c \qquad (9d)$$

$$\mathbf{n}\cdot\mathbf{v} = 0 \qquad (9e)$$

$$\delta(\mathbf{v}_t) = 0, \quad \mathbf{v}_t = (\mathbf{n} \wedge \mathbf{v}) \wedge \mathbf{n} \qquad (9f)$$

$$\delta\left[\frac{\partial \mathbf{v}_t}{\partial y} - \mathbf{v}_t \cdot \underline{k}\right] - M_T^* \nabla_s^* t - M_C^* \nabla_s^* c = 0 \qquad (9g)$$

$$\delta t = 0 \qquad (9h)$$

$$\delta c = 0 \qquad (9i)$$

where $\underline{k}$ is the non-dimensional curvature tensor: $k = \nabla_s^* \mathbf{n}$; $\mathbf{i}_x$ the unit vector of the x-axis.

The numbers therein appearing

$$Re^* = \frac{V_r}{V_\nu}, \quad Pe_T^* = \frac{V_r}{V_\alpha}, \quad Pe_C^* = \frac{V_r}{V_D} \qquad (10a)$$

$$G_T^* = \frac{V_{g_T}}{V_r}, \quad G_C^* = \frac{V_{g_C}}{V_r} \qquad (10b)$$

$$M_T^* = \frac{V_{m_T}}{V_r}, \quad M_C^* = \frac{V_{m_C}}{V_r} \tag{10c}$$

will be referred to as diffusion (10a), driving volume (10b), and driving surface (10c) parameters, respectively. The product of any driving parameter by any diffusion parameter depends only on the problem's data: its order of magnitude is uniquely determined for each specific problem.

4. ORDER OF MAGNITUDE ANALYSIS AND SIMILAR SOLUTIONS

We unify the OMA and the search for similar solutions by letting

$$\eta = y/l(x), \quad l(x) = l_0 x^{\lambda} \tag{11a}$$

$$\psi(x,y) = l_0 x^{(1-\lambda)} f(\eta) \tag{11b}$$

$$u(x,y) = x^{(1-2\lambda)} f'(\eta) \tag{11c}$$

$$v(x,y) = l_0 x^{-\lambda}[\lambda\eta f'(\eta) - (1-\lambda) f(\eta)] \tag{11d}$$

$$P(x,y) = -\tfrac{1}{2} x^{\omega} \pi(\eta) \tag{11e}$$

$$t(x,y) = -t_0 x^{\beta} g(\eta) \tag{11f}$$

$$c(x,y) = -c_0 x^{\gamma} h(\eta). \tag{11g}$$

A prime denotes derivatives with respect to the similarity variable η; $l(x)$ is the local normal scale factor and $\psi(x,y)$ the non-dimensional stream function ($\psi_y = u$; $\psi_x = -v$). The temperature and concentration scale factors, t_0 and c_0, are constants of order one. No pressure scale factor needs to be introduced since no pressure differences are imposed. The functions $f(\eta)$, $\pi(\eta)$, $g(\eta)$ and $h(\eta)$ are all of order one. The constants λ, β, γ, ω characterize the power laws assumed for the reference quantities. They will have to be determined together with the unknown constants V_r and l_0, from the OMA.

As the analysis shall be restricted to boundary layer regimes, it will always be $l_0 \ll 1$. Temperature and concentration scale factors will be used to suitably normalize the field equations in the similarity plane.

In substituting equations (11) into equations (9), we shall neglect terms in higher powers of l_0^2, and we assume that the interface maintains its hydrostatic shape (small capillary number) and that the non-dimensional principal radius of curvature of the interface, scaled by L, is at least of order one [13]. All terms involving the curvature tensor of the interface will then represent second-order effects and disappear from the first-order boundary layer equations.

Details of the substitution will be omitted and only the final results, in terms of the leading order terms, will be reported.

Field equations

$$f''' + (Re^*\, l_0^2)\{(1-\lambda)ff'' - (1-2\lambda)f'^2 - \tfrac{1}{2}[\lambda\eta\pi' - \omega\pi]x^{(\omega+4\lambda-2)}\} = -(G_T^* l_0^2)x^{(\beta+4\lambda-1)}t_0 g - (G_C^* l_0^2)x^{(\gamma+4\lambda-1)}c_0 h \tag{12a}$$

$$\pi' = 0 \tag{12b}$$

$$g'' + (Pe_T^*\, l_0^2)[(1-\lambda)fg' - \beta f'g] = 0 \tag{12c}$$

$$h'' + (Pe_C^*\, l_0^2)[(1-\lambda)fh' - \gamma f'h] = 0. \tag{12d}$$

Boundary conditions at the interface

$$f(0) = 0 \tag{13a}$$

$$-f''(0) = (M_T^*\, l_0)x^{(\beta+3\lambda-2)}t_0\beta g(0) + (M_C^*\, l_0)x^{(\gamma+3\lambda-2)}c_0\gamma h(0) \tag{13b}$$

$$t(x,0) = t_0 x^{\beta} g(0) \tag{13c}$$

$$c(x,0) = -c_0 x^{\gamma} h(0). \tag{13d}$$

The additional boundary conditions needed follow from the matching with the outer regions. We only consider here the case of a quiescent, uniform outer region. Non-quiescent outer regions [$f'(\infty) = 1$], either non-dissipative or dissipative, require a deeper and more general analysis of the asymptotic expansion techniques, applied to flow problems in liquid bridges and/or shallow cavities, which will be considered in future works.

In the subject case the asymptotic conditions are all homogeneous and read

$$f'(\infty) = g(\infty) = h(\infty) = \pi(\infty) = 0. \tag{13e}$$

Then the normal momentum equation implies $\pi(\eta) = 0$, the flow field is isobaric and the terms in the square bracket of equation (12a) disappear. For non-quiescent outer regions [$f'(\infty) = 1$], the pressure field would have the same distribution throughout the boundary layer [$\pi(\infty) = 1$], would be compatible only with an inviscid outer region and the pressure similarity exponent would be equal to $2(1-2\lambda)$.

In equations (12) there appear products of l_0^2 by diffusion and driving volume parameters; in boundary condition (13b) there appear products of l_0 by driving surface parameters. They are the diffusion measure numbers, on the left-hand side of equations (12a), (12c) and (12d) and the driving measure numbers, on the right-hand side of equations (12a) and (13b). It is seen that only driving measure numbers influence the similarity conditions.

Napolitano's constrained maximum criterion, as adapted to the subject case, requires that: all measure numbers be, at most, of order one, and the largest driving measure number be set equal to one

$$\max\{Re^*\, l_0^2, Pe_T^*\, l_0^2, Pe_C^*\, l_0^2\} \leqslant O(1) \tag{14a}$$

$$\max\{|G_T^*|\, l_0^2, |G_C^*|\, l_0^2, |M_T^*|\, l_0, |M_C^*|\, l_0\} = 1. \tag{14b}$$

The similarity conditions will then require the vanishing of the x-exponent of all, and only, the terms with measures of order one. The ratio of any two diffusion measures depends only on the ratio of the corresponding diffusion coefficients. Hence if we denote by ε the smallest diffusion coefficient and by R^* the corresponding largest diffusion parameter (with $R^* > O(1)$ since the analysis will be restricted to boundary layer regimes), criterion (14a) is satisfied by

Table 1. Different alternatives for the diffusion coefficients and corresponding orders of magnitude of the diffusion transport parameters

	Smallest diffusion coefficient		Diffusion measures	
		Momentum	Energy	Solute mass
Case	ε	$\frac{\varepsilon}{\nu}R^*l_0^2$	$\frac{\varepsilon}{\alpha}R^*l_0^2$	$\frac{\varepsilon}{D}R^*l_0^2$
a	$\varepsilon = \nu$	1	$Pr \leqslant O(1)$	$Sc \leqslant O(1)$
b	$\varepsilon = \alpha$	$\frac{1}{Pr} \leqslant O(1)$	1	$\frac{Sc}{Pr} \leqslant O(1)$
c	$\varepsilon = D$	$\frac{1}{Sc} \leqslant O(1)$	$\frac{Pr}{Sc} \leqslant O(1)$	1

letting

$$R^*l_0^2 = 1 \tag{15a}$$

where

$$R^* = \frac{V_r}{V_\varepsilon}, \quad V_\varepsilon = \frac{\varepsilon}{L}; \quad \varepsilon = \min(\nu, \alpha, D). \tag{15b}$$

The largest diffusion parameter R^* depends on the order of magnitude of the Prandtl and Schmidt numbers

$$Pr = \frac{\nu}{\alpha}, \quad Sc = \frac{\nu}{D}. \tag{16}$$

The three possible alternatives are shown in Table 1 together with the orders of magnitude of the corresponding diffusion measures.

The regions corresponding to a different order of magnitude of Prandtl and Schmidt numbers are shown in Fig. 2; the plane $O(Pr)$, $O(Sc)$ is subdivided in three main regions, corresponding to cases (a), (b) and (c) of Table 1. Our analysis falls within region (a): the smallest diffusion speed is the momentum one and the controlling transport number is the Reynolds number; the ratios of the velocity BL thickness to the temperature and solute concentration BL thicknesses are equal to the square root of the Prandtl and Schmidt numbers, respectively, which are of order of magnitude less than or equal to one.

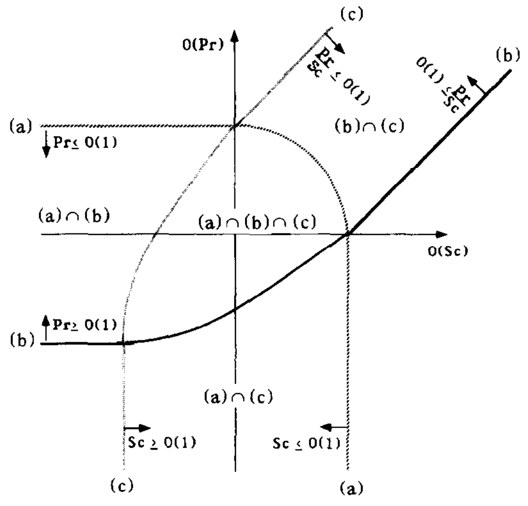


FIG. 2. Partition of the plane $O(Pr)$, $O(Sc)$ in controlling diffusion regions.

Equation (15a) is the first required relation between the unknowns V_r and l_0^2 and will be used to eliminate V_r from criterion (14b) which thus becomes

$$\max(|G_T|l_0^4, |G_C|l_0^4, |M_T|l_0^3, |M_C|l_0^3) = 1 \tag{17}$$

where the numbers

$$G_T = G_T^*R^* = \frac{V_{g_T}}{V_\varepsilon}, \quad G_C = G_C^*R^* = \frac{V_{g_C}}{V_\varepsilon} \tag{18a}$$

$$M_T = M_T^*R^* = \frac{V_{m_T}}{V_\varepsilon}, \quad M_C = M_C^*R^* = \frac{V_{m_C}}{V_\varepsilon} \tag{18b}$$

are now defined in terms of the problem's data only and will be referred to as 'characteristic' driving (volume or surface) numbers. Rigorously speaking they are all 'generalized' Reynolds numbers and there are 12 of them since each one is the ratio between one of the four driving characteristic speeds and one of the three diffusion characteristic speeds.

To propose a rational and unified terminology for these characteristic numbers is a hopeless and, perhaps, useless task. It is indeed objectively difficult to convey, in a single name, the needed three classes of information: (1) the type of driving speed (volume or surface); (2) its genesis (thermal or solutal); (3) the type of diffusion speed (momentum, energy, solute mass).

On the other hand, some names are too well rooted in classical usage to hope to change them; G_T is called the Grashof number when $\varepsilon = \nu$ and the Rayleigh number when $\varepsilon = \alpha$. Sometimes G_C is called the solutal Grashof number or solutal Rayleigh number, for $\varepsilon = \nu$ and α, respectively. Some have no name yet (e.g. G_T and G_C for $\varepsilon = D$).

We shall make a compromise by privileging the type and nature of driving speeds (the most important feature, as it will be clear shortly). Volume and surface characteristic driving numbers will be called Grashof and Marangoni numbers, respectively, whichever the type of diffusion speed involved in their definition. The nature of the driving action will be evidenced, when necessary and/or appropriate, by using the adjectives thermal or solutal.

The ratios of the two volume and surface driving measures are equal to the ratios of the corresponding driving speeds. Hence, the problem's data determine, in each specific case, the largest Grashof and Marangoni numbers, denoted by G and M, respectively, with

$$G = \left\{ \begin{array}{lll} G_T & \text{if} & \left|\dfrac{\beta_T \Delta T}{\beta_C \Delta C}\right| \geqslant O(1) \\ G_C & \text{if} & \left|\dfrac{\beta_T \Delta T}{\beta_C \Delta C}\right| \leqslant O(1) \end{array} \right\} \tag{19}$$

$$M = \left\{ \begin{array}{lll} M_T & \text{if} & \left|\dfrac{\sigma_T \Delta T}{\sigma_C \Delta C}\right| \geqslant O(1) \\ M_C & \text{if} & \left|\dfrac{\sigma_T \Delta T}{\sigma_C \Delta C}\right| \leqslant O(1) \end{array} \right\}. \tag{20}$$

Criterion (14b) can then be formulated as

$$\max(|G|l_0^4, |M|l_0^3) = 1. \tag{21}$$

Its satisfaction depends on the order of magnitude of the number $Z = |G|/|M|^{4/3}$, also known from the problem's data. If one lets

$$Z = \frac{|G|}{|M|^{4/3}} = Z_0 l_0^{\omega_1} \tag{22}$$

with Z_0 of order one, the following alternatives occur:

(a) $\omega_1 \leqslant 0$

$$\frac{|G|}{|M|^{4/3}} \geqslant O(1). \tag{23}$$

The largest driving action pertains to the volume phase; criterion (14b) requires that

$$|G|l_0^4 = 1 \tag{24}$$

and this uniquely determines the last unknown l_0. The relative measure of the surface driving action is given by

$$|M|l_0^3 = \frac{|M|}{|G|^{3/4}} = \frac{1}{Z_0^{3/4}} l_0^{-3\omega_1/4} \leqslant O(1) \tag{25}$$

as required.

(b) $\omega_1 \geqslant 0$

$$\frac{|G|}{|M|^{4/3}} \leqslant O(1). \tag{26}$$

The largest driving action belongs to the surface phase; criterion (14b) requires that

$$|M|l_0^3 = 1 \tag{27}$$

and the relative measure of the volume driving action is given by

$$|G|l_0^4 = \frac{|G|}{|M|^{4/3}} = Z_0 l_0^{\omega_1} \leqslant O(1) \tag{28}$$

as required.

The determination of V_r and l_0 is thus completed and can be usefully summarized as follows: Criteria (14a) and (14b) establish the required two relations between V_r and l_0

$$V_r = V_\varepsilon l_0^{-2} \tag{29a}$$

$$V_r = |V_d| l_0^d. \tag{29b}$$

The first one (equation (29a)) relates V_r to the smallest diffusion characteristic speed V_ε and always involves l_0^{-2}. The second one relates V_r to the largest driving characteristic speed V_d

$$|V_d| = \max(V_i, |V_m|, |V_g|) \tag{30}$$

where V_i is the imposed velocity (zero in the present case) and involves a power (d) of l_0 which depends on its type ($d = 0, 1, 2$, respectively) and not on its genesis (thermal or solutal). Solving equations (29a) and (29b) for l_0 and V_r yields

$$l_0 = \left(\frac{V_\varepsilon}{|V_d|}\right)^{1/(2+d)} = \left(\frac{1}{\mathscr{D}}\right)^{1/(2+d)}$$
$$V_r = V_\varepsilon^{d/(2+d)} |V_d|^{2/(2+d)} \tag{31}$$

where $\mathscr{D}$, the largest driving number, as seen, is the 'generalized' Reynolds number $|V_d|/V_\varepsilon$. The boundary layer thickness varies with its inverse 1/2, 1/3 and 1/4 power, respectively, for conventional boundary layers ($V_r = V_i$; $d = 0$), Marangoni boundary layers ($V_d = V_{m_T}$ or V_{m_C}; $d = 1$), natural boundary layers ($V_d = V_{g_T}$ or V_{g_C}; $d = 2$). In the 'classical' case the largest driving speed is the imposed velocity V_i and the reference velocity is just this driving speed. In the present case V_r is smaller than the driving speed, also involves the characteristic diffusion speed as it is equal to the weighted mean of the largest characteristic driving speed and the smallest characteristic diffusion speed.

The expressions for V_r and l_0 have thus been found for all the possible cases defined by the orders of magnitude of a priori known characteristic number.

In any specific case, the smallest diffusion coefficient (ε) determines the relevant largest diffusion characteristic number R and hence, the diffusion speed entering the definition of the relevant largest driving characteristic number (G or M) which in turn, determines the normal scale factor as its 1/3 and 1/4 power, respectively.

For each pertinent expression of the diffusion characteristic number there are

$$\sum_{i=1}^{4} \binom{4}{i} = 15$$

possible cases partitioned in four classes; each one grouping the $\binom{4}{i}$ cases in which (i) characteristic numbers are of the same order of magnitude.

Table 2. Classification of free convection boundary layers in terms of number and types of non-vanishing driving actions and corresponding similarity constraints. The driving number assumed to define the scale factor l_0 is circled

G		$M_T\ (\beta = 2-3\lambda)$		0	
		0	$M_C\ (\gamma = 2-3\lambda)$		0
0		$\text{\textcircled{$M_T$}}$ $(\beta = 2-3\lambda)$	$\text{\textcircled{$M_T$}}\ M_C$ $(\beta = \gamma = 2-3\lambda)$	$\text{\textcircled{$M_C$}}$ $(\gamma = 2-3\lambda)$	0
G_T $(\beta = 1-4\lambda)$	0	$G_T\ \text{\textcircled{$M_T$}}$ $\lambda = -1 \quad \forall\gamma$	$G_T\ \text{\textcircled{$M_T$}}\ M_C$ $\gamma = 1 \quad \beta = \gamma = 5$	$G_T\ \text{\textcircled{$M_C$}}$ $\beta = 1-4\lambda \quad \gamma = 2-3\lambda$	$\text{\textcircled{$G_T$}}$ $\beta = 1-4\lambda$
	G_C $(\gamma = 1-4\lambda)$	$G_T\ G_C\ \text{\textcircled{M_T}}$ $\gamma = 1 \quad \beta = \gamma = 5$	$G_T\ G_C\ \text{\textcircled{M_T}}\ M_C$ $\lambda = -1 \quad \beta = \gamma = 5$	$G_T\ G_C\ \text{\textcircled{M_C}}$ $\lambda = -1 \quad \beta = \gamma = 5$	$\text{\textcircled{$G_T$}}\ G_C$ $\beta = \gamma = 1-4\lambda$
0		$G_C\ \text{\textcircled{$M_T$}}$ $\beta = 2-3\lambda \quad \gamma = 1-4\lambda$	$G_C\ \text{\textcircled{$M_T$}}\ M_C$ $\lambda = -1 \quad \beta = \gamma = 5$	$G_C\ \text{\textcircled{$M_C$}}$ $\lambda = -1 \quad \gamma = 5 \quad \forall\beta$	$\text{\textcircled{$G_C$}}$ $\gamma = 1-4\lambda$

When two or more characteristic driving numbers are of the same order of magnitude the actual choices made in defining G or M are irrelevant on account of the introduction of the temperature and concentration scale factors t_0, c_0.

We shall adopt the following conventions:

(a) preference is always given to surface over volume driving numbers and to thermal over solutal characteristic numbers;

(b) the values of characteristic driving numbers which are not of the largest order of magnitude are simply set equal to zero since here we are only interested in first-order boundary layer theories.

According to the latter convention we have:

MBL—the two Grashof numbers are equal to zero;

NBL—the two Marangoni numbers are equal to zero;

CBL—at least one Grashof and one Marangoni number are different from zero.

The 15 possible cases are shown in Table 2. The top row represents the three types of MBLs, the left-most column the three types of NBLs. All other rows and columns represent CBLs (at least one Grashof and one Marangoni number different from zero).

In the table there are also shown: (a) the driving number assumed to define the normal scale factor l_0; (b) the similarity conditions given, in each specific case, by the sum of those indicated in the corresponding top row and left-most column.

4.1. *Determination of temperature and concentration scale factors*

The orders of magnitude one, which should have appeared in the formulation of Napolitano's criteria, have been replaced by equality to one in view of the presence of still undetermined temperature and concentration scale factors. To determine their signs and values we use the criterion of maximum normalization of the mathematical formulation (12, 13) in the similarity plane so as to minimize the number of numerical solutions needed to cover all the physically relevant cases. Furthermore, the normalization process should also take into account that the positive orientation of the variable x has been assumed to be in the sense of increasing σ or decreasing ρg_x, according to whether the surface or the volume driving action prevails.

To achieve this goal the more efficient strategy is to normalize the interface boundary conditions and use the remaining undetermined scale factors, if any, to normalize the momentum balance equation (the other two are already normalized). Temperature and concentration fields are readily normalized by setting $g(0) = h(0) = 1$.

Let, for any non-zero quantity

$$q = |q| \operatorname{sgn} q \quad \text{where} \quad \operatorname{sgn} q = \pm 1 \text{ for } q \gtrless 0. \qquad (32)$$

Upon equations (1), (11f) and (11g) it is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x} = \Delta T \sigma_T t_0 \beta x^{\beta-1} + \Delta C \sigma_C c_0 \gamma x^{\gamma-1} \qquad (33a)$$

$$\frac{\Delta\rho}{\rho_h} g_x = \Delta T \beta_T g_x t_0 g(\eta) x^{\beta} + \Delta C \beta_C g_x c_0 h(\eta) x^{\gamma} \qquad (33b)$$

where the right-hand sides are proportional to surface and volume driving actions. They will be used to determine the positive orientation of x. In the case of natural convection, the positive direction of (x) is determined, by convention, that the driving action, $(\Delta\rho/\rho_h)g_x$, is positive. In the other cases of Marangoni and combined convection, the positive direction of the axis (x) is chosen in the sense of the increasing surface tension, i.e. $\mathrm{d}\sigma/\mathrm{d}x > 0$.

On denoting by $\mathscr{L}(f)$ the left-hand side of equation (12a), the equations to be normalized can be written as

$$-\mathscr{L}(f) = A_T g x^{\beta+4\lambda-1} + A_C h x^{\gamma+4\lambda-1},$$

$$A_T = G_T l_0^4 t_0, \quad A_C = G_C l_0^4 c_0 \qquad (34a)$$

$$-f''(0) = M_T l_0^3 t_0 \beta x^{\beta+3\lambda-2} + M_C l_0^3 c_0 \gamma x^{\gamma+3\lambda-2} \qquad (34b)$$

where G_T, G_C and M_T, M_C are defined, respectively, in equations (18a) and (18b).

According to our convention, for NBLs it is $M_T = M_C = 0$ and $|G| l_0^4 = 1$. Equation (34b) is already normalized to zero and the scale factors can be used to normalize equations (34a). The relevant driving action is the volume action. The positive orientation of the x coordinate must be determined by equation (33b) which requires, upon the assumed positivity of ΔT and ΔC, that either

$$\operatorname{sgn}(\beta_T t_0 g_x) = +1 \qquad (35a)$$

or

$$\operatorname{sgn}(\beta_C c_0 g_x) = +1. \qquad (35b)$$

The constants A_T and A_C are given by

$$A_T = \operatorname{sgn}(G_T t_0) \frac{|G_T|}{|G|} |t_0| \qquad (36a)$$

$$A_C = \operatorname{sgn}(G_C c_0) \frac{|G_C|}{|G|} |c_0| \qquad (36b)$$

and, when different from zero, can be normalized to ± 1.

In all the remaining 12 cases at least one Marangoni number is different from zero so that $|M| l_0^3 = 1$, the relevant driving action is the surface action and the positive orientation must be determined by equation (33a) requiring that either one or both of the following sign relations apply:

$$\operatorname{sgn}(\sigma_T t_0 \beta) = +1, \quad \operatorname{sgn}(\sigma_C c_0 \gamma) = +1 \qquad (37a, b)$$

according to whether thermal, solutal or both Marangoni driving actions prevail.

The interface boundary condition (34b) can be normalized by putting, on account of equations (37)

$$-f''(0) = \frac{|M_T|\,|t_0|\,|\beta|}{|M|} + \frac{|M_C|\,|c_0|\,|\gamma|}{|M|} = 1 \qquad (38)$$

and the constants A_T and A_C are given by

$$A_T = \operatorname{sgn}\,(G_T t_0) \frac{|G_T|}{|M|^{4/3}} |t_0| \qquad (39a)$$

$$A_C = \operatorname{sgn}\,(G_C c_0) \frac{|G_C|}{|M|^{4/3}} |c_0| \qquad (39b)$$

unless they are equal to zero.

Whenever a Marangoni number is equal to zero the corresponding constant can be normalized to one. For instance when $M_C = 0$ and $G_C \neq 0$ the second equation (37b) does not apply, $|t_0| = 1/|\beta|$ upon equation (38) and $A_C = 1$ with

$$\operatorname{sgn}\,(c_0) = \operatorname{sgn}\,(G_C), \quad |c_0| = \frac{|M|^{4/3}}{|G_C|}. \qquad (40)$$

All possible alternatives for A_T and A_C are summarized in Table 3.

5. DISCUSSION

5.1. *Marangoni boundary layers* (*MBLs*)

The mathematical formulation of the similarity problem is

$$f''' + (1-\lambda) f f'' - (1-2\lambda) f'^2 = 0,$$

$$f(0) = f'(\infty) = 0; \quad f''(0) = -1 \qquad (41a)$$

$$g'' + Pr\,[(1-\lambda) f g' - \beta f' g] = 0, \quad g(0) = 1; g(\infty) = 0 \qquad (41b)$$

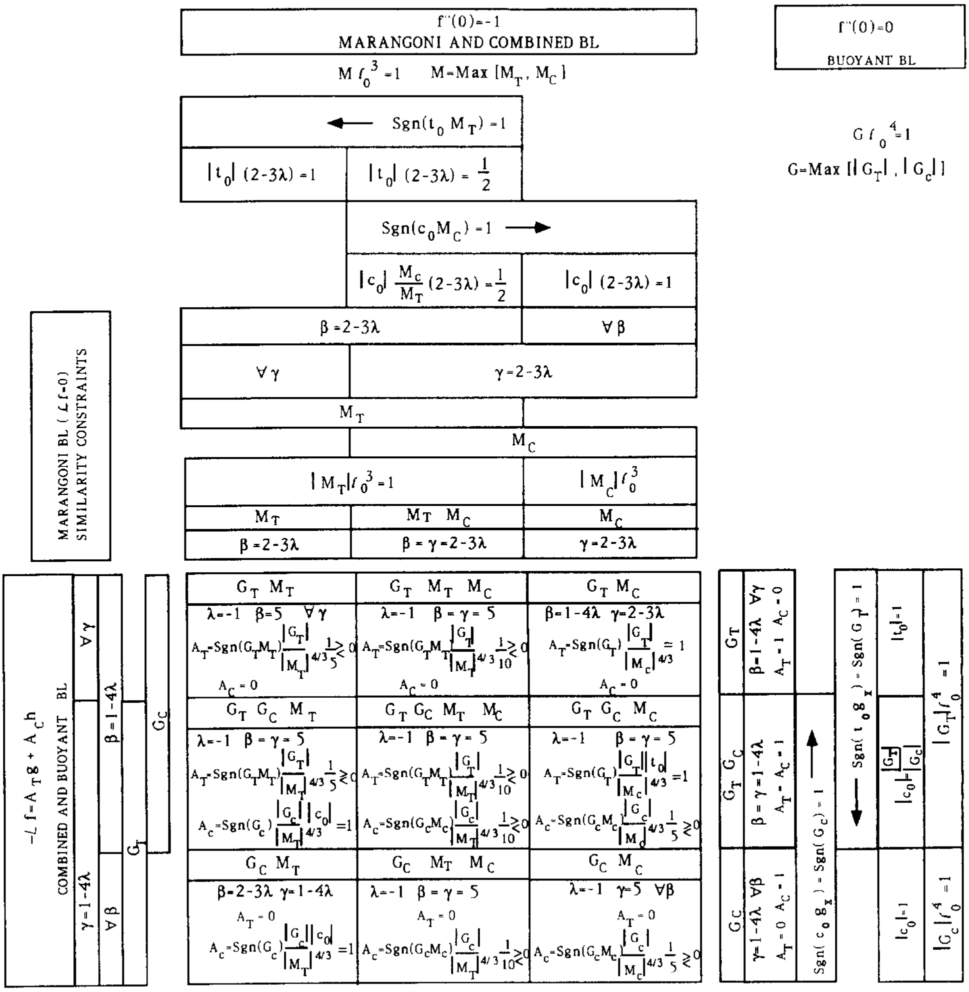
$$h'' + Sc\,[(1-\lambda) f h' - \gamma f' h] = 0, \quad h(0) = 1; h(\infty) = 0. \qquad (41c)$$

The same relation ($\beta = 2-3\lambda$; $\gamma = 2-3\lambda$) holds between λ and the parameters β and γ defining interface distribution of temperature and solutal concentration. The momentum equation is uncoupled from the energy and concentration equations and the one-third law for the boundary layer thickness applies with $|M| l_0^3 = 1$. The non-dimensional velocity field in the similar plane depends only on the parameter λ that can only have a value different from 2/3 ($\beta = \gamma = 0$, constant temperature and solutal concentration on the interface). The one parameter class of functions $f(\lambda, \eta)$ satisfying equation (41a) is thus the same as that found in ref. [2] for the single-diffusive thermal MBL. The nature of the boundary layer (thermal, solutal, double-diffusive) influences only the scale factors l_0, t_0, c_0 and establishes the same relation ($\beta = 2-3\lambda$; $\gamma = 2-3\lambda$) between λ and the parameters β and γ defining the temperature and solutal distribution on the interface. Any function $f(\lambda, \eta)$ can be interpreted as the non-dimensional stream function of either a thermal MBL with $\beta = 2-3\lambda$, or a solutal MBL for $\gamma = 2-3\lambda$, or a double-diffusive MBL for $\beta = \gamma = 2-3\lambda$. For $\lambda = 2/3$ ($\beta = 0$), $\lambda = 1/3$ ($\beta = 1$) surface gradients of temperature and/or solutal concentration vanish or are constant, respectively; for $\lambda = 0$ ($\beta = \gamma = 2$) the boundary layer thickness is constant [2]. The situation illustrated in Fig. 1 corresponds to different alternatives for σ_T, β and t_0. The temperature gradient on the interface, is given, on account of equations (8b), (11d) and (37a), by

$$T_x(x, 0) = -\operatorname{sgn}(\sigma_T) |t_0|\,|\beta| \Delta T x^{\beta-1}. \qquad (42)$$

When $\sigma_T > 0$, $T_x(x, 0)$ is negative, surface particles move from warmer to colder regions, and the hot temperature T_H is that of the upper wall. The opposite occurs for $\sigma_T < 0$: surface particles move toward warmer regions, and the hot wall is the lower one. The other two equations in problem (41) have identical structure. A two-parameter family of solutions $F(Q, \lambda)$ of the problem

Table 3. Outline of all possible alternatives for the different boundary layer regimes



$$F''+Q[(1-\lambda)fF'-(2-3\lambda)f'F]=0 \quad (43a)$$

$$F(0)=1\,;\quad F(\infty)=0 \quad (43b)$$

with $f(\eta)$ satisfying problem (41a), represents either the temperature ($Q=Pr$) or concentration ($Q=Sc$) profiles in the similarity plane.

5.1.1. *Single diffusive MBL*. For thermal MBL it is $M=M_T$; $M_C=0$ and $|M_T|\ell_0^3=1$. The temperature scale factor is given by

$$t_0=\frac{1}{|\beta|}\,\mathrm{sgn}\,(\beta\sigma_T). \quad (44)$$

Its sign is equal to that of $(\beta\sigma_T)$ and thus it is adjusted to render the surface driving force always positive for any sign of σ_T and the similarity parameter β (i.e. the sign of the scale factor t_0 is chosen in such a way that the (x) axis is always oriented in the sense of increasing surface tension). The concentration profile need not be similar. When the solute concentration follows a power law, the exponent γ can be arbitrary and the similar profile is a solution of problem (41c). The scale factor c_0 is arbitrary and can thus be normalized to ± 1. Analogous remarks apply for solutal MBL.

5.1.2. *Double-diffusive MBL*. The two Marangoni numbers are of the same order of magnitude and is still $|M_T|\ell_0^3=1$ and, as already said, $\beta=\gamma=2-3\lambda$. The two scale factors are given by (see equations (35) and (38))

$$t_0=\frac{1}{2|\beta|}\,\mathrm{sgn}\,(\beta\sigma_T),\quad c_0=\frac{1}{2|\beta|}\,\mathrm{sgn}\,(\beta\sigma_C). \quad (45a,b)$$

Their signs, once again, are chosen, consistently with our convention of $\mathrm{d}\sigma/\mathrm{d}x>0$, to account for any sign of σ_T, σ_C and of the similarity parameter β.

5.2. *Natural boundary layers* (*NBLs*)

The similarity formulation is given by equation (34), with homogeneous interface boundary conditions and the constants A_T, A_C normalized to zero or ± 1. The one-fourth boundary layer power law holds. The orientation of the x-coordinate is always in the sense of the motion, and changes in signs of the thermodynamic coefficients β_T, β_C are accounted for in the temperature and concentration scale factor t_0, c_0.

5.2.1. *Single-diffusive NBL.*

Thermal NBL. The scale factors l_0 and t_0 are given by: $l_0 = |G_T|^{-1/4}$; $t_0 = \pm 1$ with the positive sign applying for $\beta_T g_x > 0$; $A_T = 1$ and $A_C = 0$. Momentum and energy equations are coupled, and velocity and temperature profiles depend on the two parameter family of solutions of the problem

$$f''' + (1-\lambda) ff'' - (1-2\lambda) f'^2 + F = 0,$$

$$f(0) = f''(0) = f'(\infty) = 0 \qquad (46a)$$

$$F'' + Q[(1-\lambda) fF' - (1-4\lambda) f'F] = 0,$$

$$F(0) = 1; \quad F(\infty) = 0 \qquad (46b)$$

with $F = g$ and $Q = Pr$.

The concentration profiles are determined subsequently and need not be necessarily similar. If the surface distributions of solute concentration is of the power type $x^\gamma c_0 h(\eta)$ the equation to be solved is

$$h'' + Sc\,[(1-\lambda) fh' - \gamma f'h] = 0,$$

$$h(0) = 1; \quad h(\infty) = 0. \qquad (47)$$

It depends on the Schmidt number and on the arbitrary similarity parameter γ. The solute distribution is simply normalized with $c_0 = \pm 1$.

Solutal NBL. This case is the dual of the preceding one. The scale factors are given by $l_0 = |G_C|^{-1/4}$; $c_0 = \pm 1$ and the positive sign applying for $\beta_C g_x > 0$; $A_T = 0$ and $A_C = 1$. The momentum equation is coupled with the concentration equation and the problem is still mathematically described by equations (46) where now $Q = Sc$ and $F = h$. The temperature profiles can be determined subsequently and the same remarks, already made, apply.

5.2.2. *Double-diffusive NBL.*

The two Grashof numbers G_T and G_C are of the same order of magnitude, the similarity parameters are given by

$$l_0 = |G_T|^{-1/4}; \quad t_0 = \pm 1;$$

$$c_0 = \pm \frac{|G_T|}{|G_C|} = \pm \frac{|\beta_T|}{|\beta_C|} \frac{\Delta T}{\Delta C} \qquad (48)$$

where the positive sign applies for $\beta_T g_x > 0$ and $A_T = 1$; $A_C = \mathrm{sgn}\,(\beta_T \beta_C)$. The thermal buoyant characteristic speed on the interface is always positive. The intensity of the solutal buoyant characteristic velocity is measured in terms of the thermal one and normalized to one. The constant A_C measures the ratio of the two volume driving actions and is equal to plus or minus one according to whether they act in the same or in opposite directions. All three equations are coupled and the mathematical formulation of the problem reads

$$f''' + (1-\lambda) ff'' - (1-2\lambda) f'^2 + g \pm h = 0 \qquad (49a)$$

$$g'' + Pr\,[(1-\lambda) fg' - (1-4\lambda) f'g] = 0 \qquad (49b)$$

$$h'' + Sc\,[(1-\lambda) fh' - (1-4\lambda) f'h] = 0 \qquad (49c)$$

subject to the normalized boundary conditions

$$f(0) = f'(\infty) = f''(0) = 0 \qquad (50a)$$

$$g(0) = h(0) = 1 \qquad (50b)$$

$$g(\infty) = h(\infty) = 0 \qquad (50c)$$

where the plus or minus sign applies on the right-hand side of the first equation according to whether β_T and β_C have the same sign (concurrent or opposite driving actions).

5.3. *Combined boundary layers* (*CBLs*)

In the remaining nine cases both types of driving actions are present and thus they share some of the features described when dealing with MBL and NBL. We shall discuss them according to the number of driving actions (volume and surface) present.

5.3.1. *Four or three driving actions.*

In these cases, as Table 2 shows, there is only one type of similar solution characterized by the following values of the similarity parameter: $\lambda = -1$; $\beta = \gamma = 5$. Since either none or only one driving number vanishes the boundary layers always exhibit the double-diffusive character which is induced by either volume or surface driving actions. The boundary layer thickness can be indifferently described as following the 1/3 or 1/4 power law since in these cases it is always

$$\frac{|G|}{|M|^{4/3}} = O(1). \qquad (51)$$

The mathematical formulation of the similarity problem reads

$$f''' + 2ff'' - 3f'^2 + A_T g + A_C h = 0,$$

$$f(0) = f'(\infty) = 0; \quad f''(0) = -1 \qquad (51a)$$

$$g'' + Pr\,[2fg' - 5f'g] = 0, \quad g(0) = 1; g(\infty) = 0 \qquad (51b)$$

$$h'' + Sc\,[2fh' - 5f'h] = 0, \quad h(0) = 1; h(\infty) = 0 \qquad (51c)$$

where the constants A_T, A_C can be $\gtrless 0$ depending on the type of driving action which induces the double-diffusive character and can never vanish simultaneously. The case $A_T = A_C = 0$ falls within the class of MBLs for $\beta = \gamma = 5$.

5.3.2. *All four driving actions present.*

The double-diffusive character is induced by both surface and volume actions, the system (51) is completely coupled; the orientation of x is determined by the sign of G_T and it is as in Fig. 1 for $g_x \beta_T > 0$. The constants A_T and A_C are both different from zero and it is

$$A_T = \mathrm{sgn}\,(\beta_T \sigma_T) \frac{|G_T|}{|M_T|^{4/3}} \frac{1}{10},$$

$$A_C = \mathrm{sgn}\,(\beta_C \sigma_C) \frac{|G_C|}{|M_T|^{4/3}} \frac{1}{10} \qquad (52)$$

hence they are positive when the corresponding volume and surface driving actions have the same sign (concurrent).

5.3.3. *Three driving actions present.* One must distinguish if the vanishing number is a volume or a surface driving number.

Vanishing of a volume driving number. Since both M_T and M_C are different from zero, all remarks made when discussing double-diffusive MBLs apply as being specialized to the case $\beta = \gamma = 5$. In particular the 1/3 power law applies with $|M_T| l_0^3 = 1$. The only quantities to be determined are the constants A_T and A_C. The two cases $G_T = 0$, $G_C = 0$ are dual. When $G_C = 0$ it is

$$A_C = 0, \quad A_T = \pm \frac{1}{10} \frac{|G_T|}{|M_T|^{4/3}} \tag{53}$$

where the positive sign applies if $\beta_T g_x$ and σ_T have the same signs (aiding volume and surface driving actions). Momentum and energy equations are coupled. The concentration field can be subsequently determined: it must satisfy equation (51c) with $f(\eta)$ solution of the problem (51a), (51b). Analogous remarks apply for the dual case $G_T = 0$.

Vanishing of a surface driving number. Vanishing of a surface driving number allows one constant to be normalized to 1

$$M_C = 0 \Rightarrow A_C = 1 \tag{54a}$$

$$M_T = 0 \Rightarrow A_T = 1. \tag{54b}$$

The system remains completely coupled: its solution depends on three parameters: the numbers *Pr*, *Sc* and the other constant. The 1/3 power law holds with the normal scale factor l_0 determined by the non-vanishing Marangoni number, and the same considerations made when discussing single-diffusive MBLs apply. The constant different from one is given by

$$A_T = \pm \frac{1}{5} \frac{|G_T|}{|M_C|^{4/3}}, \quad A_C = \pm \frac{1}{5} \frac{|G_C|}{|M_T|^{4/3}} \tag{55a, b}$$

where the positive sign applies when $\beta_T g_x$ (respectively $\beta_C g_x$) is of the same sign as σ_C (respectively σ_T) (surface driving action aiding with the volume driving action of the other nature).

5.3.4. *One volume and one surface non-vanishing driving actions.* These four cases complete the list of 15 possible cases. Combined boundary layers of these classes share the features of the Marangoni and natural boundary layers, as neither the momentum equation nor the $f''(0)$ boundary condition are homogeneous.

Single-diffusive. Volume and surface actions are of the same nature (either thermal or solutal). The two cases are dual and, for conciseness, we shall consider the case $M_C \neq 0$ and indicate in parenthesis conditions referring to the dual case. The temperature (concentration) exponent is fixed to 5 while that of concentration (temperature) remains arbitrary. The orientation of x is determined by the sign of $\sigma_T(\sigma_C)$, and is directed as in Fig. 1 when $\sigma_T > 0$ $(\sigma_C > 0)$; the normal scale factor is equal to $|M_T|^{-1/3}$ $(|M_C|^{-1/3})$, the other parameter $t_0(c_0)$ is equal to $\pm 1/5$, where the positive sign applies for $\sigma_T > 0$ $(\sigma_C > 0)$. There is only a non-vanishing constant

$$A_T = \pm \frac{1}{5} \frac{|G_T|}{|M_T|^{4/3}} \quad \left(A_C = \pm \frac{1}{5} \frac{|G_C|}{|M_C|^{4/3}} \right) \tag{56}$$

and the positive sign applies when $\beta_T g_x(\beta_C g_x)$ and $\sigma_T(\sigma_C)$ have the same sign (concurrent volume and driving actions). Velocity and temperature (concentration) profiles are coupled and result from the solution of the system

$$f''' + 2ff'' - 3f'^2 + A_F F = 0,$$

$$f(0) = f'(\infty) = 0; \quad f''(0) = -1 \tag{57a}$$

$$F'' + Q[2fF' - 5f'F] = 0, \quad F(0) = 1; \quad F(\infty) = 0 \tag{57b}$$

where F, A_F, F, Q are equal to g, A_T, Pr (h, A_C, Sc). The solution of the concentration (temperature) field subsequently does not need to be similar and, if it is, satisfies the following problem parametered in Sc (Pr) and the arbitrary constant $\gamma(\beta)$

$$h'' + Sc\,[2fh' - \gamma f'h] = 0, \quad h(0) = 1; \quad h(\infty) = 0$$

$$(g'' + Pr\,[2fg' - \beta f'g] = 0, \quad g(0) = 1; \quad g(\infty) = 0). \tag{58}$$

The scale parameter $c_0(t_0)$ is arbitrary.

Double diffusive. Volume and surface non-vanishing driving numbers are of a different nature (one thermal and one solutal) and of the two constants A_T, A_C, one vanishes and the other is equal to one. Once again the two cases are dual and to discuss them we shall proceed as before. Both parameters β and γ are free but they are connected by a relation; for $M_T \neq 0$ $(M_C \neq 0)$ it is $4\beta - 3\gamma = 5$ $(4\gamma - 3\beta = 5)$, they are equal only for $\beta = \gamma = 5$; the orientation of the x-coordinate is determined by the sign of $\sigma_T(\sigma_C)$ and is directed as in Fig. 1 for $\sigma_T > 0$ $(\sigma_C > 0)$. The normal scale factor is $l_0 = |M_T|^{1/3}$ $(l_0 = |M_C|^{-1/3})$; the temperature (concentration) scale parameter $t_0(c_0)$ is given by

$$t_0 = \pm \frac{1}{|\beta|} \left(c_0 = \pm \frac{1}{|\gamma|} \right)$$

with the positive sign applying for $\sigma_T > 0$ $(\sigma_C > 0)$; the other scale parameter is given by

$$c_0 = \pm \frac{|M_T|^{4/3}}{|G_C|} |\beta| \quad \left(t_0 = \pm \frac{|M_C|^{4/3}}{|G_T|} |\gamma| \right)$$

and the positive sign applies when $\beta G_C M_T (\gamma G_T M_C)$ is positive; $A_C(A_T)$ vanishes and $A_T(A_C)$ is equal to one. Velocity and temperature (concentration) profiles are coupled and result from the solution of the system

$$f''' + (1-\lambda)ff'' - (1-2\lambda)f'^2 + F = 0,$$

$$f(0) = f'(\infty) = 0; \quad f''(0) = -1 \tag{59a}$$

$$F'' + Q[(1-\lambda)fF' - (2-3\lambda)f'F] = 0,$$

$$F(0) = 1\,; \quad F(\infty) = 0 \qquad (59b)$$

where for $M_C \neq 0$ ($M_T \neq 0$), F, Q, are equal to g, Pr, (h, Sc). The solutions constitute a two-parameter (λ and Q) family. As before, the solution of the concentration (temperature) field can subsequently be done and the previous remarks apply. For similar interface distributions the problem to be solved for $M_C \neq 0$ ($M_T \neq 0$) is

$$h'' + Sc\left[(1-\lambda)fh' - \frac{5+3\beta}{4}f'h\right] = 0,$$

$$h(0) = 1\,; \quad h(\infty) = 0$$

$$\left(g'' + Pr\left[(1-\lambda)fg' - \frac{5+3\gamma}{4}f'g\right] = 0,\right.$$

$$\left. g(0) = 1\,; \quad g(\infty) = 0\right) \qquad (60)$$

for arbitrary non-vanishing $\beta(\gamma)$.

6. NUMERICAL RESULTS

The field equations have been solved via a quasi-linearization method [14] and numerical results obtained for each type of boundary layer (Marangoni, natural, combined) when driving actions, either of the same nature (single-diffusive BL) or of a different nature (double-diffusive BL), are present. Since local (interface velocity, heat and mass transfer bulk coefficients) and global (displacement thickness) properties on the interface are expressed in terms of the f, g, and h derivatives evaluated at $\eta = 0$, and of the asymptotic value $f(\infty)$, the numerical values obtained for these quantities have been tabulated for various Prandtl and Schmidt numbers and different values of the similarity parameter.

The dimensional interfacial velocity $U(X, 0)$ is proportional to the f derivative $f'(0)$

$$U(X,0) = V_r x^{(1-2\lambda)} f'(0). \qquad (61)$$

Heat and mass transfer bulk coefficients are expressed in terms of local non-dimensional parameters, namely the Nusselt and Sherwood numbers, defined respectively as

$$Nu_{,x} = -\frac{(\mathbf{n}\cdot\nabla T)_{y=0}}{\Delta T}x = \frac{t_0}{l_0}x^{(\beta-\lambda+1)}g'(0) \qquad (62a)$$

$$Sh_{,x} = -\frac{(\mathbf{n}\cdot\nabla C)_{y=0}}{\Delta C}x = \frac{c_0}{l_0}x^{(\gamma-\lambda+1)}h'(0) \qquad (62b)$$

so that they are proportional to the g and h derivatives $g'(0)$, $h'(0)$. The displacement thickness, defined as

$$\delta^* = l_0 x^\lambda \int_0^\infty \frac{f'(\eta)}{f'(0)}\,\mathrm{d}\eta = l_0 x^\lambda \frac{f(\infty)}{f'(0)} \qquad (63)$$

is proportional to the ratio $f(\infty)/f'(0)$.

In the case of the Marangoni boundary layers (see Section 5.1) the nature of the boundary layer (thermal, solutal, double-diffusive) does not influence the resulting mathematical problem, which consists of the two-point non-linear problem (41a) for the function $f(\eta)$ and problem (43) for the temperature and the concentration in the similarity plane. Numerical solutions of this problem have been already found in ref. [2], whose results, for different values of the similarity parameter ($\beta = 0.5, 1, 2$) and various Prandtl numbers, can be also applied to the double-diffusive case ($F(\eta) = g(\eta)$ or $F(\eta) = h(\eta)$ according to whether $Q = Pr$ or Sc) by considering that the similarity variable (η_0) and the corresponding function $f_0(\eta_0)$, introduced in the above quoted reference, are linked to η, $f(\eta)$ (present work) by the transformation

$$\eta = (1-\lambda)^{-1/3}\eta_0 \qquad (63a)$$

$$f(\eta) = (1-\lambda)^{-2/3}f_0(\eta_0). \qquad (63b)$$

Table 4 reports numerical values of $f'(0)$ and $f(\infty)$ for β ranging from 1 to 10. In the case $\beta = 2$ (constant boundary layer thickness) there is an exact solution of problem (41a): $f(\eta) = 1 - e^{\eta}$.

Tables 5(a) and (b) summarize the results obtained in the case of natural (Table 5(a)) and combined (Table 5(b)) single-diffusive boundary layers (equations for $f(\eta)$ and $F(\eta)$ coupled, with $F(\eta)$ equal to $g(\eta)$ or to $h(\eta)$).

Table 5(a) shows the numerical values of $f'(0)$, $g'(0)$ and $f(\infty)$ for various Prandtl numbers and different values of the parameter β, in the case of the thermal NBL. The dual case of solutal NBL can be formally obtained by substituting the Prandtl number (Pr) with the Schmidt number (Sc), $g(\eta)$ with $h(\eta)$ and β with γ. The effect of increasing Prandtl number, at given β, results in the reduction of the velocity level and in the thinning of the thermal boundary layer, whereas heat transfer at the interface increases. In the case of combined Marangoni and natural convection the effect of aiding surface tension and buoyancy driven flows results in increasing interface velocity and heat transfer, compared to the flow driven by Marangoni or by natural convection only.

The results shown in Table 5(b) coincide with those obtained in ref. [5], if one considers that the similarity variable (η_0) and the function $f_0(\eta_0)$ considered in the above reference are linked to η, $f(\eta)$ (present work) by the transformation

$$\eta = 2^{-1/3}\eta_0 \qquad (64a)$$

$$f(\eta) = 2^{-2/3}f_0(\eta_0). \qquad (64b)$$

In the case of double-diffusive natural boundary layers (see Section 5.2) all three equations (momentum, energy and concentration) are coupled and the solutions constitute a three-parameter (λ, Pr, Sc) family. The numerical results corresponding to the two cases of aiding and opposing thermal and solutal driving actions are reported in Tables 6(a) and 7(a), respectively.

When the actions (thermal and solutal) are aiding,

Table 4. Numerical values of $f'(0)$ and $f(\infty)$ for different values of the similarity parameter β, for MBLs

β	$f'(0)$	$f(\infty)$
1	1.296	1.481
2	1.000	1.000
3	0.864	0.783
4	0.782	0.656
5	0.723	0.570
6	0.679	0.508
7	0.644	0.460
8	0.615	0.423
9	0.591	0.392
10	0.570	0.366

$f'''+(1-\lambda)ff''-(1-2\lambda)f'^2 = 0$, $\lambda = (2-\beta)/3$, with boundary conditions: $f(0) = 0$, $f'(\infty) = 0$, $f''(0) = -1$.

$h(\eta, \lambda,\ Pr = a,\ Sc = b)$ is equal to $g(\eta, \lambda,\ Pr = b,\ Sc = a)$, so that the quantitative investigation has been performed for the Prandtl number ranging from 0.1 to 10 with $Sc \geqslant Pr$ (these values cover a wide class of liquid mixtures of interest). Numerical values of $f'(0)$, $-g'(0)$, $-h'(0)$, $f(\infty)$ calculated in the case of aiding thermal and solutal actions are considerably higher than those corresponding to the case of opposing actions. Increasing the Schmidt number, at given Pr and β, confines the buoyancy effect, due to solute concentration differences, to a thinner solutal boundary layer: therefore, velocity and heat transfer decrease for aiding thermal and solutal driving actions (Table 6(a)) and increase for opposing actions (Table 7(a)), whereas mass transfer increases vs Schmidt in both cases. The effect of Prandtl number is the same found for single-diffusive BL: increasing Prandtl, heat transfer increases, but interface velocity and mass transfer are reduced. Typical profiles of the functions $f'(\eta)$, $g(\eta)$ and $h(\eta)$ in the similarity plane are reported in Figs. 3–5 for double-diffusive NBLs with different values of the parameter β, with $Pr = 0.7$ and $Sc = 7$.

The case $\beta = -3/5$ corresponds to the rising plume flow considered by Gebhart and Pera [4], whose results can be found from our calculations, for this particular value of β, applying the following transformations

$$\eta = 4^{-1/4}\left(1+\frac{G_T}{G_C}\right)\eta_0 \tag{65a}$$

$$f(\eta) = 4^{3/4}\left(1+\frac{G_T}{G_C}\right)f_0(\eta_0) \tag{65b}$$

$$g(\eta) = g_0(\eta_0) \tag{65c}$$

$$h(\eta) = \frac{G_T}{G_C}\,h_0(\eta_0) \tag{65d}$$

where the subscripts (0) refer to Gebhart and Pera variables. In this case ($\lambda = 2/5$) the solution for $g(\eta)$ and $h(\eta)$ is

$$g(\eta) = \mathrm{e}^{-3/5\,Pr\,F(\eta)} \tag{66a}$$

$$h(\eta) = \mathrm{e}^{-3/5\,Sc\,F(\eta)} \tag{66b}$$

$$F(\eta) = \int_0^\eta f(\eta)\,\mathrm{d}\eta;\quad g'(0) = h'(0) = 0. \tag{66c}$$

Table 5. Numerical values of $f'(0)$, $g'(0)$ and $f(\infty)$ for different Prandtl numbers, at different β, for thermal natural (a) and combined (b) boundary layers

(a)

	$\beta = 1$			$\beta = 5$			$\beta = 10$		
Pr	$f'(0)$	$-g'(0)$	$f(\infty)$	$f'(0)$	$-g'(0)$	$f(\infty)$	$f'(0)$	$-g'(0)$	$f(\infty)$
0.1	0.833	0.312	2.666	0.488	0.486	1.450	0.361	0.586	0.993
0.5	0.703	0.674	1.523	0.413	1.04	0.788	0.306	1.254	0.524
1	0.633	0.926	1.179	0.373	1.425	0.606	0.276	1.712	0.402
5	0.471	1.859	0.788	0.278	2.832	0.405	0.206	3.396	0.268
10	0.408	2.472	0.698	0.241	3.756	0.359	0.179	4.502	0.238
15	0.374	2.912	0.654	0.221	4.419	0.337	0.164	5.294	0.224

$f'''+(1-\lambda)ff''-(1-2\lambda)f'^2+g = 0$, $g''+Pr\,[(1-\lambda)fg'-\beta f'g] = 0$, $\lambda = (1-\beta)/4$, with boundary conditions: $f(0) = 0$, $f'(\infty) = 0$, $f''(0) = 0$, $g(0) = 1$, $g(\infty) = 0$.

(b)

Pr	$f'(0)$	$-g'(0)$	$f(\infty)$
0.1	0.944	0.563	1.545
0.5	0.884	1.356	0.881
1	0.853	1.971	0.724
5	0.794	4.587	0.611
10	0.776	6.541	0.597
15	0.768	8.038	0.592

$f'''+2ff''-3f'^2+g = 0$, $g''+Pr\,[2fg'-5f'g] = 0$, with boundary conditions: $f(0) = 0$, $f'(\infty) = 0$, $f''(0) = -1$, $g(0) = 1$, $g(\infty) = 0$.

Table 6. Numerical values of $f'(0)$, $g'(0)$, $h'(0)$ and $f(\infty)$ for different Prandtl and Schmidt numbers at different β, for double-diffusive natural (a) and combined (b) boundary layers with aiding thermal and solutal driving actions

(a)

		$\beta = 1$				$\beta = 5$				$\beta = 10$			
Pr	Sc	$f'(0)$	$-g'(0)$	$-h'(0)$	$f(\infty)$	$f'(0)$	$-g'(0)$	$-h'(0)$	$f(\infty)$	$f'(0)$	$-g'(0)$	$-h'(0)$	$f(\infty)$
0.1	0.1	1.183	0.371	0.371	3.694	0.690	0.579	0.579	1.938	0.513	0.696	0.696	1.307
	0.2	1.148	0.356	0.542	3.310	0.670	0.559	0.837	1.728	0.496	0.675	1.000	1.169
	0.5	1.096	0.341	0.876	3.149	0.641	0.538	1.337	1.650	0.477	0.647	1.597	1.127
	1	1.056	0.333	1.243	3.102	0.617	0.525	1.886	1.641	0.459	0.632	2.251	1.111
	2	1.017	0.327	1.760	3.076	0.595	0.516	2.646	1.612	0.440	0.622	3.161	1.094
	5	0.971	0.322	2.793	3.055	0.567	0.506	4.120	1.601	0.420	0.610	4.926	1.087
	8	0.951	0.320	3.432	3.047	0.555	0.503	5.168	1.596	0.411	0.606	6.179	1.084
	10	0.942	0.319	3.823	3.044	0.550	0.501	5.756	1.594				
	15					0.541	0.499	7.000	1.592				
1	2	0.847	1.061	1.560	1.298	0.499	1.637	2.382	0.658	0.370	1.967	2.857	0.442
	5	0.792	1.019	2.441	1.245	0.467	1.574	3.075	0.630	0.346	1.892	4.438	0.424
	8	0.768	1.003	3.061	1.230	0.453	1.548	4.638	0.623	0.336	1.861	5.554	0.419
	10	0.758	0.996	3.407	1.224	0.446	1.537	5.159	0.621				
	12	0.749	0.990	3.720	1.220	0.442	1.529	5.629	0.618	0.327	1.838	6.739	0.416
	15	0.740	0.984	4.139	1.216	0.436	1.519	6.262	0.615	0.323	1.826	7.496	0.414
10	12	0.566	2.910	3.199	0.823	0.334	4.422	4.858	0.411				
	15	0.553	2.875	3.548	0.810	0.327	4.369	5.383	0.404	0.248	5.30	5.82	0.280
	20	0.538	2.833	4.054	0.796	0.318	4.306	6.146	0.397	0.242	5.237	6.449	0.275
	30	0.519	2.780	4.895	0.779	0.306	4.225	7.415	0.388				

$f'''+(1-\lambda)ff''-(1-2\lambda)f'^2+g+h=0$, $g''+Pr\,[(1-\lambda)fg'-\beta f'g]=0$, $h''+Sc\,[(1-\lambda)fh'-\gamma f'h]=0$, $\lambda=(1-\beta)/4$, $\gamma=\beta$, with boundary conditions: $f(0)=0, f'(\infty)=0, f''(0)=0, g(0)=1, g(\infty)=0, h(0)=1, h(\infty)=0$.

(b)

Pr	Sc	$f'(0)$	$-g'(0)$	$-h'(0)$	$f(\infty)$
0.1	0.1	1.090	0.638	0.638	1.981
	0.2	1.076	0.619	0.946	1.787
	0.5	1.050	0.599	1.567	1.712
	1	1.030	0.588	2.273	1.691
	2	1.014	0.581	3.271	1.680
	5	0.994	0.574	5.250	1.672
	8	0.986	0.572	6.673	1.669
	10	0.982	0.571	7.475	1.668
	15	0.976	0.570	9.180	1.667
1	2	0.932	2.081	3.078	0.757
	5	0.909	2.040	4.960	0.737
	8	0.899	2.028	6.326	0.732
	10	0.895	2.020	7.090	0.730
	12	0.892	2.018	7.780	0.728
	15	0.888	2.013	8.714	0.727
10	12	0.821	6.743	7.419	0.610
	15	0.817	6.720	8.310	0.607
	20	0.812	6.701	9.617	0.604

$f'''+2ff''-3f'^2+g+h=0$, $g''+Pr\,[2fg'-5f'g]=0$, $h''+Sc\,[2fh'-5f'h]=0$, with boundary conditions: $f(0)=0$, $f'(\infty)=0$, $f''(0)=-1$, $g(0)=1$, $g(\infty)=0$, $h(0)=1$, $h(\infty)=0$.

For $\beta > -3/5, g'(0) < 0, h'(0) < 0$, i.e. for $t_0 > 0$ and $c_0 > 0$, energy and mass of solute diffuse from the liquid to the gas; for $\beta = -3/5$, $g'(0) = 0$, $h'(0) = 0$, i.e. the interface is adiabatic and with zero mass transfer; for $\beta < -3/5, g'(0) > 0, h'(0) > 0$, i.e. energy and mass of solute diffuse toward the liquid. The same features are shown in Figs. 6 and 7, where numerical values of $f'(0)$ and $-g'(0)$ are plotted vs the Schmidt number for $Pr = 0.7$ and different values of β. For $\beta = -3/5$, $g'(0)$ is identically equal to zero, for $\beta < -3/5$ (respectively $\beta > -3/5$) it is positive (respectively negative) and increases (respectively decreases) for increasing Schmidt numbers.

Results for the more complex case of double-diffusive combined boundary layers are shown in Tables 6(b) and 7(b) corresponding to aiding and opposing thermal and solutal driving actions, respectively. Since in this case all possible driving actions are present (volume and surface; thermal and solutal), the results obtained confirm many features already described when dealing with the cases discussed before. In particular, the effect of increasing Schmidt number is

Table 7. Numerical values of $f'(0)$, $g'(0)$, $h'(0)$ and $f(\infty)$ for different Prandtl and Schmidt numbers at different β, for double-diffusive natural (a) and combined (b) boundary layers with opposing thermal and solutal driving actions

(a)

		$\beta = 1$				$\beta = 5$				$\beta = 10$			
Pr	Sc	$f'(0)$	$-g'(0)$	$-h'(0)$	$f(\infty)$	$f'(0)$	$g'(0)$	$h'(0)$	$f(\infty)$	$f'(0)$	$-g'(0)$	$-h'(0)$	$f(\infty)$
0.1	0.2	0.275	0.213	0.305	2.362	0.160	0.325	0.461	1.270	0.118	0.388	0.550	0.876
	0.5	0.434	0.259	0.599	2.782	0.251	0.396	0.897	1.458	0.185	0.474	1.071	0.993
	2	0.594	0.288	1.382	2.930	0.346	0.445	2.072	1.533	0.256	0.535	2.474	1.042
	5	0.669	0.297	2.305	2.967	0.390	0.462	3.459	1.553	0.288	0.555	4.132	1.055
	8	0.699	0.301	2.975	2.979	0.407	0.467	4.467	1.559	0.301	0.562	5.337	1.059
1	2	0.278	0.653	0.935	0.983	0.164	0.992	1.414	0.490	0.121	1.189	1.692	0.332
	5	0.408	0.779	1.791	1.090	0.240	1.189	2.707	0.548	0.178	1.426	3.239	0.370
	8	0.453	0.815	2.387	1.115	0.267	1.247	3.606	0.560	0.198	1.495	4.315	0.378
	12	0.484	0.839	3.022	1.130	0.285	1.283	4.566	0.568	0.211	1.540	5.464	0.384
	15	0.499	0.849	3.431	1.136	0.294	1.300	5.183	0.572	0.218	1.560	6.202	0.386
10	12	0.109	1.304	1.431	0.393	0.064	1.978	2.169	0.180	0.048	2.369	2.598	0.134
	15	0.106	1.577	1.937	0.475	0.095	2.391	2.936	0.224	0.070	2.865	3.516	0.162
	20	0.205	1.779	2.529	0.532	0.121	2.698	3.831	0.255	0.090	3.233	4.588	0.181
	30	0.249	1.959	3.420	0.579	0.147	2.972	5.177	0.280	0.109	3.560	6.200	0.197

$f''' + (1-\lambda)ff'' - (1-2\lambda)f'^2 + g - h = 0$, $g'' + Pr[(1-\lambda)fg' - \beta f'g] = 0$, $h'' + Sc[(1-\lambda)fh' - \gamma f'h] = 0$, $\lambda = (1-\beta)/4$, $\gamma = \beta$, with boundary conditions: $f(0) = 0$, $f'(\infty) = 0$, $f''(0) = 0$, $g(0) = 1$, $g(\infty) = 0$, $h(0) = 1$, $h(\infty) = 0$.

(b)

Pr	Sc	$f'(0)$	$-g'(0)$	$-h'(0)$	$f(\infty)$
0.1	0.2	0.770	0.476	0.730	1.459
	0.5	0.813	0.517	1.333	1.591
	2	0.864	0.544	2.993	1.637
	5	0.889	0.552	4.945	1.647
	8	0.889	0.555	6.358	1.651
1	2	0.760	1.831	2.722	0.664
	5	0.791	1.888	4.600	0.692
	8	0.803	1.907	5.950	0.699
	12	0.811	1.920	7.397	0.702
	15	0.815	1.925	8.328	0.704
10	12	0.728	6.312	6.947	0.564
	15	0.733	6.337	7.838	0.567
	20	0.738	6.365	9.142	0.571
	30	0.745	6.398	11.329	0.583

$f''' + 2ff'' - 3f'^2 + g - h = 0$, $g'' + Pr[2fg' - 5f'g] = 0$, $h'' + Sc[2fh' - 5f'h] = 0$, with boundary conditions: $f(0) = 0$, $f'(\infty) = 0$, $f''(0) = -1$, $g(0) = 1$, $g(\infty) = 0$, $h(0) = 1$, $h(\infty) = 0$.

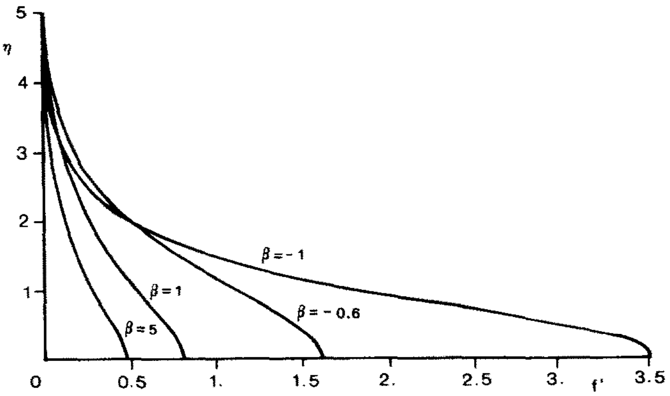


FIG. 3. Profiles of $f'(\eta)$ in the similarity plane for natural boundary layers (NBLs) at different values of the similarity parameter β, when aiding thermal and solutal driving actions are present. The Prandtl and Schmidt numbers are equal to $Pr = 0.7$, $Sc = 7$.

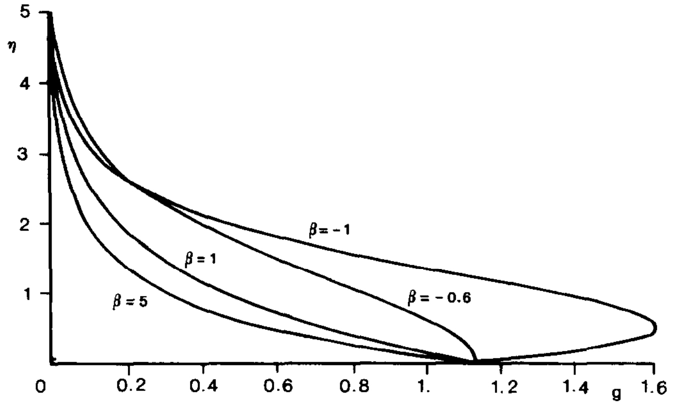


FIG. 4. Profiles of $g(\eta)$ in the similarity plane for natural boundary layers (NBLs) at different values of the similarity parameter β, when aiding thermal and solutal driving actions are present. The Prandtl and Schmidt numbers are equal to $Pr = 0.7$, $Sc = 7$.

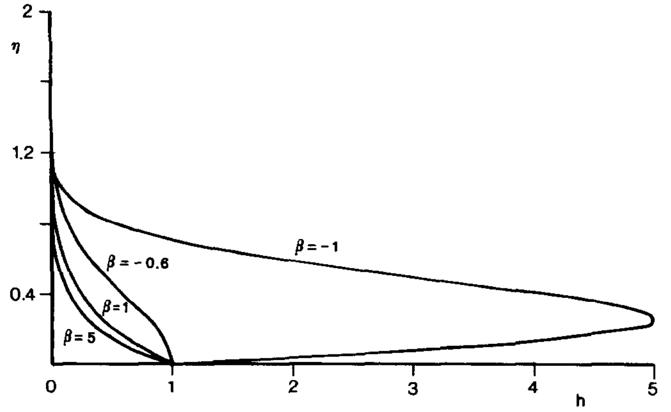


FIG. 5. Profiles of $h(\eta)$ in the similarity plane for natural boundary layers (NBLs) at different values of the similarity parameter β, when aiding thermal and solutal driving actions are present. The Prandtl and Schmidt numbers are equal to $Pr = 0.7$, $Sc = 7$.

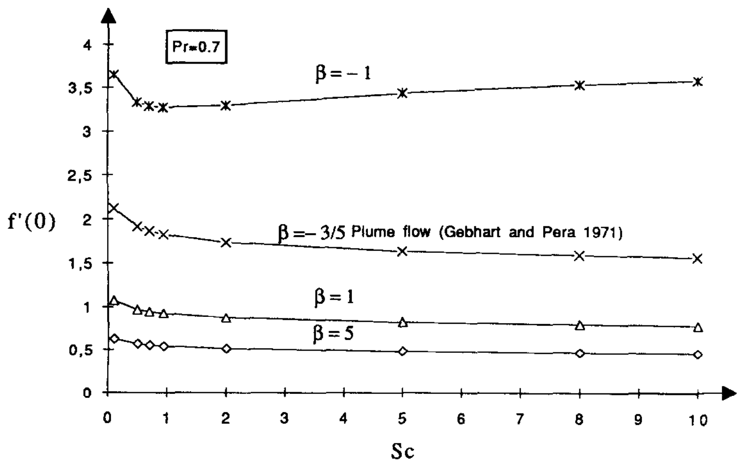


FIG. 6. Profiles of $f'(0)$ vs Schmidt number for natural boundary layers (NBLs) at different values of the similarity parameter β, when aiding thermal and solutal driving actions are present. The Prandtl number is equal to $Pr = 0.7$.

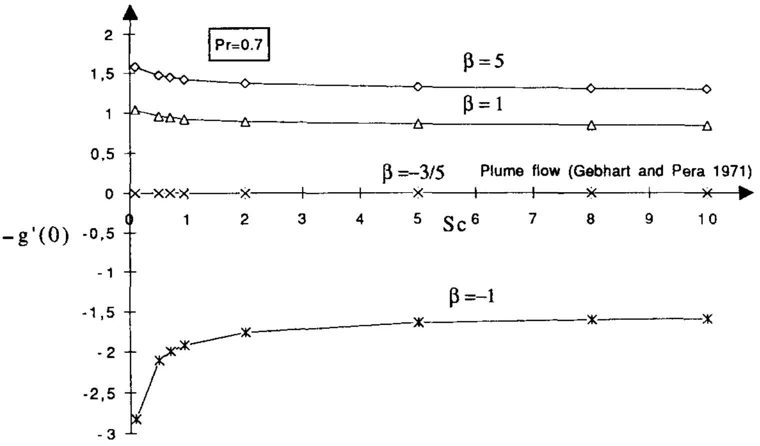


FIG. 7. Profiles of $-g'(0)$ vs Schmidt number for natural boundary layers (NBLs) at different values of the similarity parameter β, when aiding thermal and solutal driving actions are present. The Prandtl number is equal to $Pr = 0.7$.

seen to be the reduction or the increasing of interface velocity and heat transfer, for aiding or opposing thermal and solutal driving actions, respectively, whereas the solutal boundary layer thickness decreases and solute mass transfer increases. At given Sc and β, velocity levels and solute mass diffusion are reduced by increasing the Prandtl number, whereas heat transfer increases. Combined convection corresponds to higher velocities, heat and mass transfer, when compared to the case of pure Marangoni or natural convection, due to the effect of aiding surface tension and buoyancy driven flows. Finally, velocity, heat and mass transfer, in the case of aiding thermal and solutal actions, are considerably higher than those corresponding to the case of opposing actions.

The values of $f'(0)$, corresponding to CBL and to the analogous case of NBL with $\beta = \gamma = 5$ are compared in Figs. 8 and 9, where $f'(0)$ is plotted vs Schmidt and various Prandtl numbers, in the case of aiding (Fig. 8) and opposing (Fig. 9) actions, respectively. For any value of Sc, $f'(0)$ (i.e. the interfacial velocity) for CBL is higher than for NBL, and, for each type of BL, attains higher values in the case of aiding thermal and solutal effects.

This behaviour is summarized in Fig. 10, concerning CBL and NBL for $Pr = 0.1$, in both situations of opposing and aiding thermal and solutal driving actions; $f'(0)$ increases for opposing and decreases for aiding thermal and solutal driving actions, since the species diffusion buoyancy effects, as Sc increases, is confined more and more near the interface region (and thus the corresponding solutal driving action is reduced).

Analogous considerations apply to the values of the g derivative $g'(0)$ (related to the local Nusselt parameter). Figures 11 and 12 show that $g'(0)$

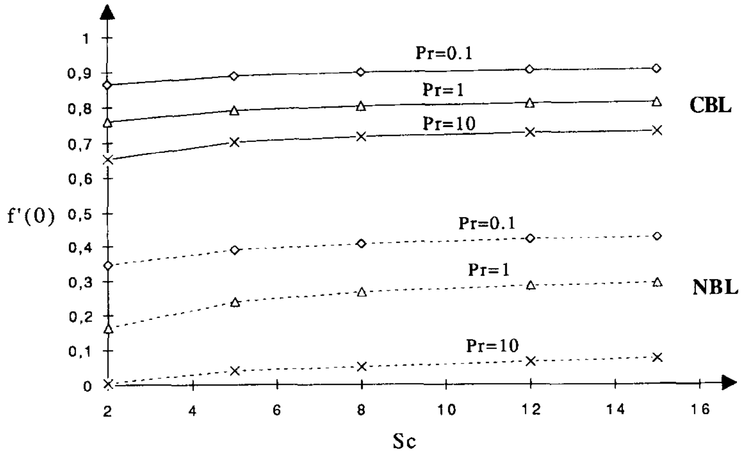


FIG. 8. Profiles of $f'(0)$ vs Schmidt number for natural (NBLs) and combined (CBLs) boundary layers at various Prandtl numbers with aiding thermal and solutal driving actions. In the case of NBLs the exponents of the temperature and concentration are equal to 5.

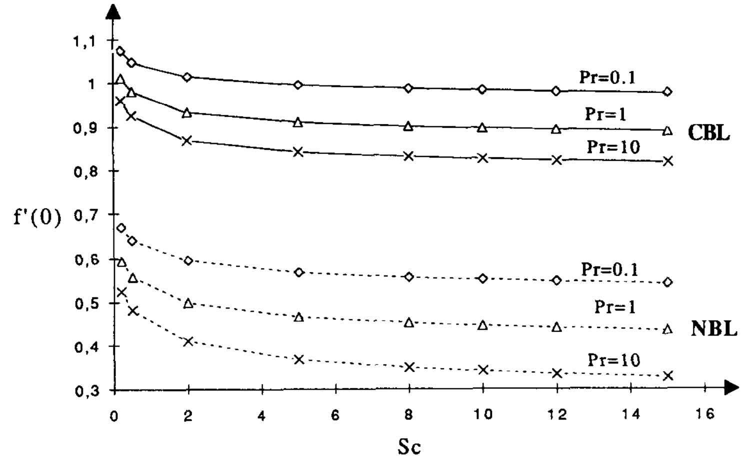


FIG. 9. Profiles of $f'(0)$ vs Schmidt number for natural (NBLs) and combined (CBLs) boundary layers at various Prandtl numbers with opposing thermal and solutal driving actions. In the case of NBLs the exponents of the temperature and concentration are equal to 5.

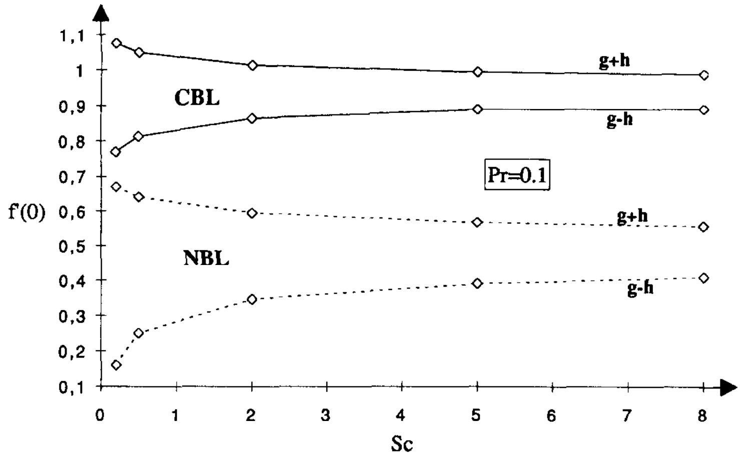


FIG. 10. Profiles of $f'(0)$ vs Schmidt number for natural (NBLs) and combined (CBLs) boundary layers with aiding $(g+h)$ and opposing $(g-h)$ thermal and solutal driving actions. In the case of NBLs the exponents of the temperature and concentration are equal to 5. The Prandtl number is equal to $Pr = 0.1$.

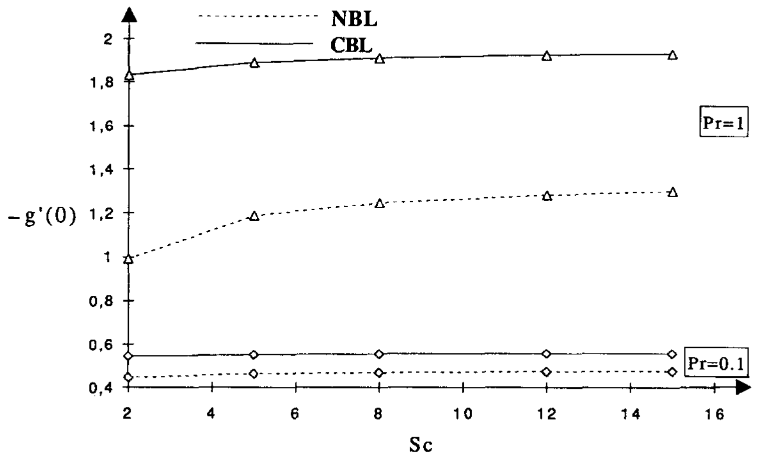


FIG. 11. Profiles of $-g'(0)$ vs Schmidt number for natural (NBLs) and combined (CBLs) boundary layers at various Prandtl numbers with aiding thermal and solutal driving actions. In the case of NBLs the exponents of the temperature and concentration are equal to 5.

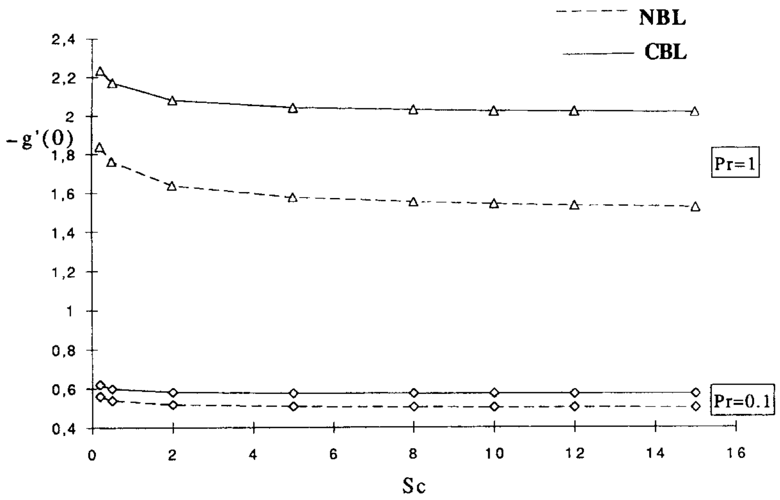


FIG. 12. Profiles of $-g'(0)$ vs Schmidt number for natural (NBLs) and combined (CBLs) boundary layers at various Prandtl numbers with opposing thermal and solutal driving actions. In the case of NBLs the exponents of the temperature and concentration are equal to 5.

increases or decreases vs Schmidt number according to whether the solutal and thermal driving actions are aiding or opposing, and the heat transfer is higher for CBL than for NBL.

The mass transfer, expressed by the h derivative $h'(0)$ (related to the local Sherwood parameter), increases vs Schmidt number in both cases of aiding and opposing thermal and solutal actions (Figs. 13 and 14).

The influence of the Prandtl number on the flow and transport properties at the interface is shown in the diagrams of Figs. 8–14. As the Prandtl number increases, heat transfer increases (Figs. 11 and 12), whereas the interfacial velocity (Figs. 8 and 9) and the mass transfer (Figs. 13 and 14) decrease since thermal buoyancy effect, at high Prandtl numbers, is confined more deeply near the interface. The effect of volume driving numbers (thermal and solutal) is shown in Table 8, where, for given Prandtl and Schmidt numbers ($Pr = 1$; $Sc = 5$) numerical results are reported for A_T and A_C (coefficients in equations (51)) ranging from 0 to 1, in the case of double-diffusive CBL.

Velocity, heat and mass transfer are enhanced by increasing the thermal (A_T) or the solutal (A_C) volume driving action. The case $A_C = 0$ represents the situation in which only one volume driving action (the thermal one) is present, whereas for $A_T = A_C = 1$ the thermal and solutal driving forces have the same order of magnitude.

The higher values of the solute mass transfer par-

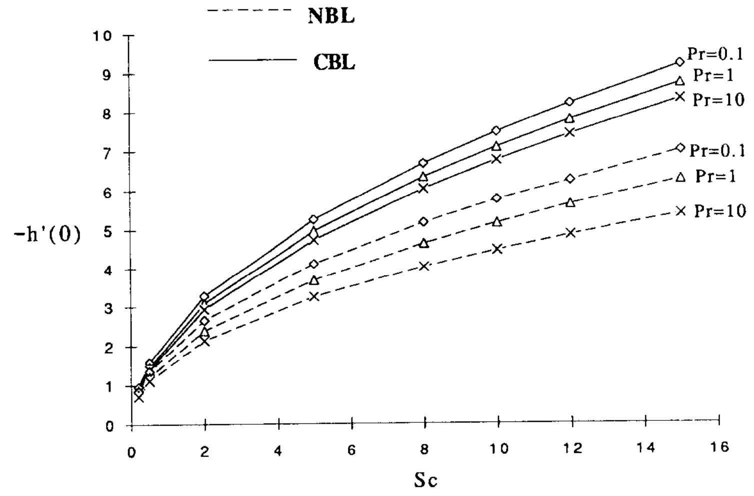


FIG. 13. Profiles of $-h'(0)$ vs Schmidt number for natural (NBLs) and combined (CBLs) boundary layers at various Prandtl numbers with aiding thermal and solutal driving actions. In the case of NBLs the exponents of the temperature and concentration are equal to 5.

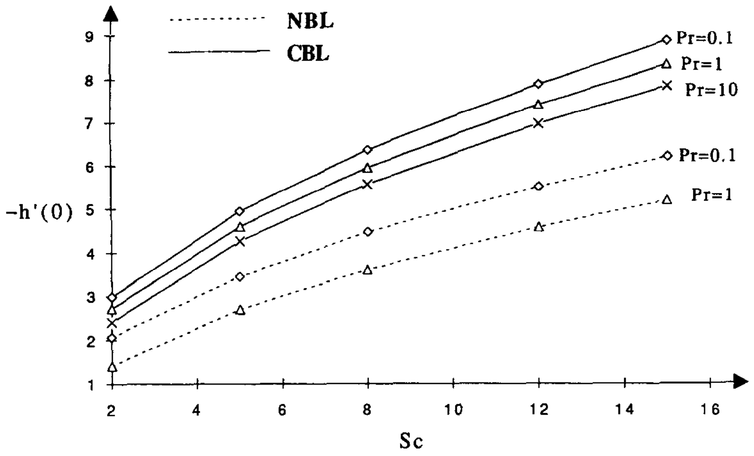


FIG. 14. Profiles of $-h'(0)$ vs Schmidt number for natural (NBLs) and combined (CBLs) boundary layers at various Prandtl numbers with opposing thermal and solutal driving actions. In the case of NBLs the exponents of the temperature and concentration are equal to 5.

ameter $[-h'(0)]$, with respect to the heat transfer parameter $[-g'(0)]$, are related to the higher value of Sc with respect to Pr. However, as discussed before, the roles of Pr and Sc, A_T and A_C, can be interchanged in interpreting the results, by interchanging correspondingly the roles played by the temperature and solute concentration fields. With respect to this, it is important to remember that the present analysis has been limited to Prandtl and Schmidt numbers of order of magnitude less than or equal to one. Further quantitative investigations will be needed to cover the other regions of the plane (Pr, Sc) (see Fig. 2), where Pr and/or Sc are of order of magnitude greater than one and the relevant transport numbers are $(Re\,Pr)$ or $(Re\,Sc)$ according to the pertinent expression of the diffusion characteristic speed (thermal or solutal). The analysis of these regimes will be the object of future works.

Table 8. Numerical values of $f'(0)$, $g'(0)$, $h'(0)$ and $f(\infty)$ for different A_T and A_C, for double-diffusive combined boundary layers

		$Pr = 1$, $Sc = 5$			
A_T	A_C	$f'(0)$	$-g'(0)$	$-h'(0)$	$f(\infty)$
0.25	0	0.760	1.815	4.481	0.624
	0.25	0.778	1.840	4.539	0.632
	0.5	0.794	1.864	4.594	0.640
	0.75	0.810	1.887	4.647	0.648
	1	0.826	1.909	4.697	0.655
0.5	0	0.794	1.874	4.598	0.664
	0.25	0.810	1.896	4.650	0.671
	0.5	0.825	1.918	4.701	0.677
	0.75	0.841	1.938	4.749	0.684
	1	0.855	1.958	4.795	0.690
0.75	0	0.825	1.925	4.702	0.696
	0.25	0.840	1.946	4.751	0.703
	0.5	0.855	1.965	4.794	0.708
	0.75	0.869	1.984	4.842	0.714
	1	0.883	2.002	4.885	0.719
1	0	0.853	1.971	4.798	0.726
	0.25	0.868	1.990	4.843	0.729
	0.5	0.882	2.008	4.886	0.735
	0.75	0.896	2.026	4.928	0.740
	1	0.909	2.040	4.960	0.745

$f'''+2ff''-3f'^2+A_Tg+A_Ch = 0$, $g''+Pr[2fg'-5f'g] = 0$, $h''+Sc[2fh'-5f'h] = 0$, with boundary conditions: $f(0) = 0, f'(\infty) = 0, f''(0) = -1, g(0) = 1, g(\infty) = 0$.

7. CONCLUSIONS AND FINAL COMMENTS

A unified approach to free convection (natural, Marangoni, combined) boundary layers along vertical free surfaces has been presented when the (volume and surface) driving actions are due to volume differences and surface gradients of temperature and solute concentration.

For each class of free convection similar solutions have been derived and characterized, on the basis of a rigorous order of magnitude analysis. For each pertinent expression of the diffusion characteristic number (depending on the orders of magnitude of the Prandtl and Schmidt numbers) the 15 possible BL regimes have been partitioned in four classes, each one grouping the $\binom{4}{i}$ cases in which (i) characteristic driving numbers are of the same order of magnitude. The similarity conditions, the pertinent expressions for the scale factors and the field equations have been specified and discussed for each type of BL.

An important and original contribution of the work is represented by the mathematical formulation in the similarity plane, which has been devoted to the normalization of the interface boundary conditions and of the balance equations, in order to minimize the number of numerical solutions needed to cover all the physically relevant cases. In this way, all possible alternatives for the numerical values of the driving numbers influence only the scale factors and not the field equations, so that the quantitative investigation can be performed by solving only one problem for each type of BL.

In the particular case of MBLs, it has been found that the mathematical formulation of the similarity problem for a single-diffusive (thermal or solutal) BL is the same as for a double-diffusive BL. Any function $f(\lambda, \eta)$ can be interpreted as the non-dimensional stream function of either a thermal or a solutal BL with equal driving actions, or of a double-diffusive BL if the thermal and solutal driving actions are opportunely weighted by means of the temperature and concentration scale factors.

Natural boundary layers have been investigated by considering only the problem (49), (50), rather than solving the field equations for different values of the Grashof numbers as performed, for instance, by Gebhart and Pera [4] in the case of double-diffusive boundary layers along rigid walls.

For each type of BL we have obtained velocity, temperature and concentration profiles in the similarity plane; flow and transport properties at the liquid–gas interface (interfacial velocity, heat and mass transfer bulk coefficients) have been given for a wide range of Prandtl and Schmidt numbers and different values of the similarity parameter.

Even if the present study is focused on plane boundary layers along vertical liquid–gas interfaces, the formulation developed here can be extended to a BL along horizontal free surfaces, either for uncoupled or for coupled flow fields of the two interfacing fluids (i.e. for liquid–gas and liquid–liquid configurations). In this work temperature and concentration on the free surface have been assumed to be known power functions of the (x) coordinate, and for the asymptotic conditions at the infinite, the simplest case of uniform, quiescent outer regions has been considered. Next works will be devoted to the study of more complex coupled situations (flow problems in liquid bridges and/or shallow cavities), where the distributions of the thermofluid dynamic properties on the interface are unknown, and the flow field variables are assigned only in the initial section $x = 0$ and at the infinite (non-quiescent outer regions, either non-dissipative or dissipative). Other problems of the theory of MBLs to be investigated include unsteady flows, corner boundary layers, as well as the stability of the already studied MBLs.

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COUCHES LIMITES DOUBLEMENT DIFFUSIVES LE LONG DE SURFACES VERTICALES LIBRES

Résumé—On traite la convection naturelle doublement diffusive (ou thermosolutale), c'est-à-dire convection due aux forces de flottement (convection naturelle) et gradients de tension de surface (convection de Marangoni), avec différences de volume, de température et de concentration de soluté aux surfaces. L'attention est portée sur les couches limites formées le long d'une interface verticale liquide–gaz, lorsque les nombres de transport adimensionnels caractéristiques sont suffisamment grands, de façon que le flottement et la tension interfaciale conduisent des écoulements dans des cavités ouvertes différentiellement chauffées et des ponts liquides. Des classes de solutions affines sont obtenues pour chaque classe de convection sur la base d'une analyse rigoureuse d'ordre de grandeur. Des profils de vitesse, de température et de concentration sont décrits dans le plan de similarité; les propriétés d'écoulement et de transport à l'interface liquide–gaz sont obtenus pour un large domaine de nombres de Prandtl et de Schmidt et différentes valeurs du paramètre de similarité.

DOPPELT-DIFFUSIVE GRENZSCHICHTEN ENTLANG EINER SENKRECHTEN FREIEN OBERFLÄCHE

Zusammenfassung—Die vorliegende Arbeit behandelt freie Konvektion bei gekoppeltem Wärme- und Stoffübergang. Treibende Kräfte sind dabei Auftriebskräfte (natürliche Konvektion) und Oberflächenspannungsgradienten (Marangoni-Konvektion), die durch Temperatur- und Konzentrationsgradienten in den Phasengrenzflächen hervorgerufen werden. Die Untersuchung konzentriert sich auf Grenzschichten, die sich entlang einer senkrechten Gas/Flüssigkeits-Phasengrenze ausbilden, wenn die angemessen definierten charakteristischen dimensionslosen Transportgrößen groß genug werden. Beispiele für derartige Grenzschichten sind Strömungen in differentiell beheizten, offenen Vertiefungen und an Flüssigkeitsbrücken, die von Auftriebs- und Oberflächenkräften angetrieben werden. Klassen von ähnlichen Lösungen für jede Situation werden mittels rigoroser Größenordnungsanalyse bestimmt. Geschwindigkeits-, Temperatur- und Konzentrationsprofile in der Ähnlichkeitsebene werden dargestellt. Für einen großen Wertebereich der Prandtl- und der Schmidt-Zahl sowie für verschiedene Werte des Ähnlichkeitsparameters werden die Strömungs- und Transportparameter bestimmt: Geschwindigkeit in der Phasengrenze, Gesamtwärme- und -stoffübergangskoeffizienten.

ТЕПЛОВЫЕ И ДИФФУЗИОННЫЕ ПОГРАНИЧНЫЕ СЛОИ ВДОЛЬ ВЕРТИКАЛЬНЫХ СВОБОДНЫХ ПОВЕРХНОСТЕЙ

Аннотация—Исследущтся диффузия тепла и массы при комбинированной свободной конвекции, т.е. конвекции за счет подъемных сил (естественная конвекция) и за счет градиентов поверхностного натяжения (конвекция Марангони), обусловленных разностью объемов и поверхностными градиентами температур и концентраций растворенного вещества. Основное внимание обращается на формирование пограничных слоев вдоль вертикальной границы раздела жидкость–газ при достаточно больших безразмерных числах, характеризующих конвективные течения, в незамкнутых нагреваемых полостях и жидких мостиках. Для каждого вида конвекции получены классы подобных решений. Представлены профили скоростей, температур и концентраций в плоскости подобия; получены характеристики течения и переноса на границе раздела жидкость–газ (скорость на границе раздела и объемные коэффициенты тепло- и массопереноса) в широком диапазоне изменений чисел Прандтля и Шмидта при различных значениях критерия подобия.